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Advances in Mathematics 239 (2013) 109–127

ADVANCES IN Mathematics

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On common fundamental domains

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Received 3 September 2009; accepted 22 February 2013
Available online 22 March 2013

Communicated by Kenneth Falconer

Abstract

We find conditions under which two measure preserving actions of two groups on the same space have a common fundamental domain. Our results apply to commuting actions with separate fundamental domains, lattices in groups of polynomial growth, and some semidirect products. We prove that two lattices of equal co-volume in a group of polynomial growth, one acting on the left, the other on the right, have a common fundamental domain.
Published by Elsevier Inc.

MSC: 22F10; 22D05; 22D30; 05B45; 05B40

Keywords: Fundamental domain; Packing; Group action; Lattice; Polynomial growth

Contents

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0001-8708/$ - see front matter. Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.aim.2013.02.016

1. Introduction

In [8], motivated by the study of Weyl–Heisenberg (or Gabor) frames, Deguang Han and Yang Wang proved that two lattices in $\mathbb{R}^n$ having the same finite co-volume have a common measurable fundamental domain. We will present a much more general result in Theorem 1.9: consider two lattices in a group of polynomial growth (Definition 3.2), one acting on the left and the other acting on the right. Assuming that the two given lattices have the same co-volume, we then prove that they must have a common measurable fundamental domain.

It is easy to see that the condition is necessary, i.e., that if there is a common fundamental domain for a given left/right pair of lattices, then the value of the co-volume numbers computed from the two sides must be the same. But the converse implication seems unexpected: It states that a difference in these two numbers is the only obstruction. In other words, when a pair of lattices is given, then a difference in the value of these co-volume numbers is the only obstruction to the existence of a common fundamental domain.

Since our fundamental domains and the corresponding tilings of the ambient group are defined within the measurable category, there is a vast variation of possibilities, and it is often difficult to produce algorithms for computing common fundamental domains. While lattices are known in the case of $\mathbb{R}^n$, this is not the case for non-abelian groups. Hence in Section 3 we specialize to nilpotent Lie groups, and the Heisenberg group $G = H(\mathbb{R})$ in particular. Moreover for each lattice, we show that there are natural and concrete choices of fundamental domains. Nonetheless, explicit formulas for *common* fundamental domains are hard to come by.

Despite these difficulties, our condition in Theorem 1.9 that a pair of left/right lattices yields equal co-volume numbers is relatively easy to verify. And hence we get the existence of a common fundamental domain in all these cases.

Fundamental domains are important in direct integral decompositions for unitary representations (see e.g., [12]) where one often use fundamental domains as “parameters” in direct integral decompositions. Hence for such applications, it is important that a measurable choice be made. In our discussion below of existence of a common fundamental domains, measurability is understood implicitly.

Common fundamental domains appear also in connection with measure equivalence of groups, a notion introduced by Gromov in [7]. Two groups Γ and Λ are said to be *measure equivalent* if there exists a measure space (Ω, m) and two commuting measure preserving actions of Γ and Λ such that each action has a fundamental domain. Such a measure space is called a measure equivalence coupling of the two groups. For a survey on the measure equivalence of groups we refer to [5]. We mention here just a few remarkable results: any two amenable groups are measure equivalent; on the other hand if Γ is a lattice in a higher rank simple Lie group, and if a countable group Λ is measure equivalent (ME) to Γ, then Λ itself must essentially be a lattice in a higher rank simple Lie group (see [5] and the references therein).

In [4] it is proved that two groups have orbit equivalent actions iff they have a measure equivalent coupling, where the fundamental domains have equal measure, iff they have a measure equivalent coupling with a *common* fundamental domain. Thus the situation we are interested in