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Advances in Mathematics 239 (2013) 164–189

ADVANCES IN Mathematics

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Generalized limits with additional invariance properties and their applications to noncommutative geometry

Fedor Sukochev*, Alexandr Usachev, Dmitriy Zanin

School of Mathematics and Statistics, University of New South Wales, Sydney, 2052, Australia

Received 15 July 2011; accepted 26 February 2013
Available online 25 March 2013

Communicated by Alain Connes

Abstract

Let $\mathbb{N}$ be the set of all natural numbers and $l_\infty = l_\infty(\mathbb{N})$ (resp., $L_\infty = L_\infty(0, \infty)$) be the Banach space of all bounded sequences $x = (x_1, x_2, \ldots)$ (resp., (classes of) Lebesgue measurable bounded functions on $(0, \infty)$) with the uniform norm. We work with subsets of the set of all normalized positive functionals on l_∞ and L_∞ which are used in noncommutative geometry to define various classes of Dixmier traces (singular positive traces) on the ideal $\mathcal{M}_{1,\infty}$ of compact operators on an infinite-dimensional Hilbert space H with logarithmic divergence of the partial sums of their singular values. These classes of traces are of importance in noncommutative geometry and with each such a class (say $\mathcal{B}$) of traces, a subset $\mathcal{M}_{1,\infty}^{\mathcal{B}} \subset \mathcal{M}_{1,\infty}$ on which all traces from $\mathcal{B}$ coincide is linked. The main objective of the present article is to give new characterizations of $\mathcal{M}_{1,\infty}^{\mathcal{B}}$ for commonly occurring classes of Dixmier traces. We also answer a question from Benameur and Fack (2006) [4] concerning conditions on a Dixmier trace under which Lidskii-type formulae from that article hold.

MSC: primary 58B34; 46L52

Keywords: Dixmier traces; Measurable elements; Generalized limits; Cesàro operator

* Corresponding author.
E-mail addresses: f.sukochev@unsw.edu.au (F. Sukochev), a.usachev@unsw.edu.au (A. Usachev), d.zanin@unsw.edu.au (D. Zanin).

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http://dx.doi.org/10.1016/j.aim.2013.02.012

1. Introduction

This paper studies an important concept of measurable operators introduced in noncommutative geometry by A. Connes in [9] and [10, Definition 7, IV.2.β]. We begin by describing most frequently used classes of Dixmier traces which play a role in the definition of this concept.

Consider the Banach ideal $(\mathcal{M}_{1,\infty}, \|\cdot\|_{1,\infty})$ of compact operators in the algebra $B(H)$ of all bounded linear operators on a separable Hilbert space H given by

$$\mathcal{M}_{1,\infty} := \left\{ T : \|T\|_{1,\infty} := \sup_{n\in\mathbb{N}} \frac{1}{\log(1+n)} \sum_{k=1}^{n} \mu_k(T) < \infty \right\}.$$

Here $\mu_n(T)$, $n \in \mathbb{N}$ is the n-th singular value of a compact operator T.

Most of the results in the present paper formulated for the ideal $\mathcal{M}_{1,\infty}$ in the algebra $B(H)$. However, they are valid for an arbitrary semifinite von Neumann algebra.

A normalized positive linear functional on l_∞ which equals the ordinary limit on convergent sequences is called a generalized limit. For an arbitrary translation and dilation invariant generalized limit ω on l_∞ (precise definition of this class can be found in the next section), J. Dixmier [11] proved that the weight

$$\mathrm{Tr}_\omega(T) := \omega\left(\left\{ \frac{1}{\log(1+n)} \sum_{k=1}^{n} \mu_k(T) \right\}_{n=1}^{\infty} \right), \quad 0 \le T \in \mathcal{M}_{1,\infty},$$

extends to a non-normal trace (a Dixmier trace) on $\mathcal{M}_{1,\infty}$. There are several natural subclasses of Dixmier traces in noncommutative geometry. First, it was observed by A. Connes [10], that in order to ensure that Tr_ω is a positive linear functional on $\mathcal{M}_{1,\infty}$, it is sufficient to only assume that ω is a dilation invariant generalized limit on l_∞. Later (see e.g. [24]), the class $\mathcal{D}$ of all traces generated by such dilation invariant ω's was termed Dixmier traces and we shall adopt here this terminology. A natural way to generate such ω's was suggested by A. Connes [10, Section IV, 2β] by observing that for any generalized limit γ on l_∞ a functional $\omega := \gamma \circ M$ is dilation invariant. Here, the bounded operator $M : l_\infty \to l_\infty$ is given by the formula

$$(Mx)_n = \frac{1}{\log(1+n)} \sum_{k=1}^{n} \frac{x_k}{k}, \quad n \ge 1.$$

Finally, various important formulae of noncommutative geometry, like Connes' formula for a representative of the Hochschild class of the Chern character for (p, ∞)-summable spectral triples (see e.g. [5, Theorem 7] and [4, Theorem 6]) as well as those involving heat kernel estimates and generalized ζ-functions (see e.g. [4–7,36]) are frequently established for yet a smaller subset $\mathcal{D}_M$ of Dixmier traces, when the generalized limit ω is assumed to be M-invariant, that is when

$$\omega = \omega \circ M.$$

For brevity, we sometimes refer to the class $\mathcal{D}_M$ as the class of all M-invariant Dixmier traces.

Definition 1. Let $\mathcal{B}$ denote the class $\mathcal{D}$ or $\mathcal{D}_M$. The set $\mathcal{M}_{1,\infty}^{\mathcal{B}}$ of all $\mathcal{B}$-measurable elements consists of all elements $x \in \mathcal{M}_{1,\infty}$ such that $\mathrm{Tr}_\omega(x)$ takes the same value for all $\mathrm{Tr}_\omega \in \mathcal{B}$.

We refer the reader to [2,8,10,13,25,31] for discussion of properties and concrete examples of $\mathcal{D}$-measurable elements, although we point out that the precise relationship between these classes