Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 238 (2013) 1–23

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Polynomiality of monotone Hurwitz numbers in higher genera[☆]

I.P. Goulden[a], Mathieu Guay-Paquet[b,*], Jonathan Novak[c]

[a] *Department of Combinatorics & Optimization, University of Waterloo, Canada*
[b] *LaCIM, Université du Québec à Montréal, Canada*
[c] *Department of Mathematics, Massachusetts Institute of Technology, USA*

Received 12 October 2012; accepted 21 January 2013
Available online 5 March 2013

Communicated by Ravi Vakil

Abstract

Hurwitz numbers count branched covers of the Riemann sphere with specified ramification, or equivalently, transitive permutation factorizations in the symmetric group with specified cycle types. Monotone Hurwitz numbers count a restricted subset of these branched covers, related to the expansion of complete symmetric functions in the Jucys–Murphy elements, and have arisen in recent work on the asymptotic expansion of the Harish-Chandra–Itzykson–Zuber integral. In previous work we gave an explicit formula for monotone Hurwitz numbers in genus zero. In this paper we consider monotone Hurwitz numbers in higher genera, and prove a number of results that are reminiscent of those for classical Hurwitz numbers. These include an explicit formula for monotone Hurwitz numbers in genus one, and an explicit form for the generating function in arbitrary positive genus. From the form of the generating function we are able to prove that monotone Hurwitz numbers exhibit a polynomiality that is reminiscent of that for the classical Hurwitz numbers, *i.e.*, up to a specified combinatorial factor, the monotone Hurwitz number in genus g with ramification specified by a given partition is a polynomial indexed by g in the parts of the partition.

MSC: primary 05A15; 14E20; secondary 15B52

Keywords: Hurwitz numbers; Matrix models; Enumerative geometry

☆ IPG and MGP were supported by NSERC.
* Corresponding author.
E-mail addresses: ipgoulden@uwaterloo.ca (I.P. Goulden), mathieu.guaypaquet@lacim.ca (M. Guay-Paquet), jnovak@math.mit.edu (J. Novak).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2013.01.012

Contents

1. Introduction

1.1. Classical Hurwitz numbers

Hurwitz numbers count branched covers of the Riemann sphere with specified ramification data. The most general case which is commonly studied is that of *double* Hurwitz numbers $H_g(\alpha, \beta)$, where two points on the sphere are allowed to have non-simple ramification. That is, for two partitions $\alpha, \beta \vdash d$, the Hurwitz number $H_g(\alpha, \beta)$ counts degree d branched covers of the Riemann sphere by Riemann surfaces of genus g with ramification type α over 0 (say), ramification type β over ∞ (say), and simple ramification over r other arbitrary but fixed points (where $r = 2g - 2 + \ell(\alpha) + \ell(\beta)$ by the Riemann–Hurwitz formula), up to isomorphism. The original case of *single* Hurwitz numbers $H_g(\alpha)$ is obtained by taking $\beta = (1^d)$, corresponding to having no ramification over ∞.

If we label the preimages of some unbranched point by $1, 2, \ldots, d$, then Hurwitz's monodromy construction [11] identifies $H_g(\alpha, \beta)$ bijectively with the number of $(r + 2)$-tuples $(\rho, \sigma, \tau_1, \ldots, \tau_r)$ of permutations in the symmetric group $\mathbf{S}_d$ such that

(1) ρ has cycle type α, σ has cycle type β, and the τ_i are transpositions;
(2) the product $\rho\sigma\tau_1 \cdots \tau_r$ is the identity permutation;
(3) the subgroup $\langle \rho, \sigma, \tau_1, \ldots, \tau_r \rangle \subseteq \mathbf{S}_d$ is transitive; and
(4) the number of transpositions is $r = 2g - 2 + \ell(\alpha) + \ell(\beta)$.

The double Hurwitz numbers were first studied by Okounkov [16], who addressed a conjecture of Pandharipande [17] in Gromov–Witten theory by proving that a certain generating function for these numbers is a solution of the 2-Toda lattice hierarchy from the theory of integrable systems. Okounkov's result implies that a related generating function for the single