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On elliptic equations in a half space or in convex wedges with irregular coefficients

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Abstract

We consider second-order elliptic equations in a half space with leading coefficients measurable in a tangential direction. We prove the W^2_p-estimate and solvability for the Dirichlet problem when $p \in (1, 2]$, and for the Neumann problem when $p \in [2, \infty)$. We then extend these results to equations with more general coefficients, which are measurable in a tangential direction and have small mean oscillations in the other directions. As an application, we obtain the W^2_p-solvability of elliptic equations in convex wedge domains or in convex polygonal domains with discontinuous coefficients.

Keywords: Second-order elliptic equations; Boundary value problems; Measurable coefficients; Sobolev spaces

1. Introduction

In this paper we study second-order elliptic equations in non-divergence form:

$$Lu - \lambda u = f$$

in a half space, where $\lambda \geq 0$ is a constant and

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu$$

is an uniformly elliptic operator with bounded and measurable coefficients. The leading coefficients a^{ij} are symmetric, merely measurable in a tangential direction, and either independent or

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have very mild regularity in the orthogonal directions. This type of equations typically arises in homogenization of layered materials with boundaries perpendicular to the layers.

The L_p theory of non-divergence form second-order elliptic and parabolic equations with discontinuous coefficients was studied extensively by many authors. According to the well-known counterexample of Nadirashvili, in general there does not exist solvability theory for uniformly elliptic operators with general bounded and measurable coefficients. In the last fifty years, many efforts were made to treat particular types of discontinuous coefficients. The W_2^2-estimate for elliptic equations with measurable coefficients in smooth domains was obtained by Bers and Nirenberg [3] in the two dimensional case in 1954, and by Talenti [42] in any dimensions under the Cordes condition. In [5] Campanato established the W_p^2-estimate for elliptic equations with measurable coefficients in 2D for p in a neighborhood of 2. A corresponding result for parabolic equations can be found in Krylov [26]. By using explicit representation formulas, Lorenzi [32,33] studied the W_2^2 and W_p^2 estimates for elliptic equations with piecewise constant coefficients in the upper and lower half spaces. See also [39] for a similar result for parabolic equations and a recent paper [21] by Kim for elliptic equations in $\mathbb{R}^d$ with leading coefficients which are discontinuous at finitely many parallel hyperplanes. In [8] Chiti obtained the W_2^2-estimate for elliptic equations in $\mathbb{R}^d$ with coefficients which are measurable functions of a fixed direction.

Another notable type of discontinuous coefficients contains functions with vanishing mean oscillation (VMO) introduce by Sarason. The study of elliptic and parabolic equations with VMO coefficients was initiated by Chiarenza, Frasca, and Longo [6] in 1991 and continued in [7,4]. We also refer the read to Lieberman [31] for an elementary treatment of elliptic equations with VMO coefficients in Morrey spaces, and Acerbi and Mingione [1] for p-Laplacian type parabolic systems with VMO coefficients. In [27] Krylov gave a unified approach to investigating the L_p solvability of both divergence and non-divergence form elliptic and parabolic equations in the whole space with leading coefficients that are in VMO in the spatial variables (and measurable in the time variable in the parabolic case). Unlike the arguments in [6,7,4] which are based on the certain representation formulas and Calderón–Zygmund theory, the proofs in [27] rely mainly on pointwise estimates of sharp functions of spatial derivatives of solutions, so that VMO coefficients are treated in a rather straightforward manner.[1] Later this approach was further developed to treat more general types of coefficients. In [22] Kim and Krylov established the W_p^2-estimate, for $p \in (2, \infty)$, of elliptic equations in $\mathbb{R}^d$ with leading coefficients measurable in a fixed direction and VMO in the orthogonal directions, which, in particular, generalized the result in [8]. By using a standard method of odd and even extensions, their result carries over to equations in a half space when the leading coefficients are measurable in the normal direction and VMO in all the tangential directions. Recently, the results in [22] were extended in [29,12], in the latter of which the restriction $p > 2$ was dropped. We also mention that in [11] the W_p^2-estimates, for $p \geq 2$ close to 2, were obtained for elliptic and parabolic equations. The leading coefficients are assumed to be measurable in the time variable and *two* coordinates of space variables, and almost VMO with respect to the other coordinates. In particular, these results extended the aforementioned results in [5,26] to high dimensions.

Since the work in [22], the following problem remains open: *Do we have a W_p^2-estimate for uniformly elliptic operators in a half space with leading coefficients measurable in a tangential direction and, say, with the homogeneous Dirichlet boundary condition?*

[1] See also [18,10] for earlier work.