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Butterflies in a semi-abelian context

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Abstract

It is known that monoidal functors between internal groupoids in the category *Grp* of groups constitute the bicategory of fractions of the 2-category $Grpd(Grp)$ of internal groupoids, internal functors and internal natural transformations in *Grp*, with respect to weak equivalences (that is, internal functors which are internally fully faithful and essentially surjective on objects). Monoidal functors can be equivalently described by a kind of weak morphisms introduced by B. Noohi under the name of butterflies. In order to internalize monoidal functors in a wide context, we introduce the notion of internal butterflies between internal crossed modules in a semi-abelian category $\mathcal{C}$, and we show that they are morphisms of a bicategory $\mathcal{B}(\mathcal{C})$. Our main result states that, when in $\mathcal{C}$ the notions of Huq commutator and Smith commutator coincide, then the bicategory $\mathcal{B}(\mathcal{C})$ of internal butterflies is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences.

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Keywords: Butterfly; Internal groupoid; Bicategory of fractions; Weak equivalence; Semi-abelian category

Contents

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1. Introduction

A groupoid in the category of groups is a special case of strict monoidal category, tensor product being provided by the group structure on objects and arrows. Therefore, beyond internal functors, as arrows between groupoids in groups we can consider monoidal functors, that is, functors between the underlying categories

$$F: \mathbb{H} \to \mathbb{G},$$

equipped with a natural and coherent family of isomorphisms

$$F^{x,y}: F_0(x) + F_0(y) \to F_0(x+y) \quad x, y \in H_0.$$

Both notions of monoidal functor and internal functor are relevant as morphisms of groupoids in groups (just to cite an example, as a special case of monoidal functors we get group extensions, whereas in the same case internal functors give split extensions, see Section 7, so the question of expressing monoidal functors in an internal way arises.

Three progresses have been recently accomplished in this direction. In [40] (see also [41] and [2]) B. Noohi has proved that the bicategory having groupoids in groups as objects and monoidal functors as 1-cells can be equivalently described using crossed modules of groups as objects and what he calls butterflies as arrows. Moreover, in a paper with E. Aldrovandi [2], the