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Prime ends for domains in metric spaces

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Abstract

In this paper we propose a new definition of prime ends for domains in metric spaces under rather general assumptions. We compare our prime ends to those of Carathéodory and Näkki. Modulus ends and prime ends, defined by means of the p-modulus of curve families, are also discussed and related to the prime ends. We provide characterizations of singleton prime ends and relate them to the notion of accessibility of boundary points, and introduce a topology on the prime end boundary. We also study relations between the prime end boundary and the Mazurkiewicz boundary. Generalizing the notion of John domains, we introduce almost John domains, and we investigate prime ends in the settings of John domains, almost John domains and domains which are finitely connected at the boundary.

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Contents

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1. Introduction

The classical Dirichlet boundary value problem associated with a differential operator L consists in finding a function u which satisfies the equation $Lu = 0$ in a domain Ω and the boundary condition $u = f$ on $\partial\Omega$ for given boundary data $f : \partial\Omega \to \mathbb{R}$. This problem has been studied extensively for various elliptic differential operators, including the Laplacian Δ and its nonlinear counterpart the p-Laplacian Δ_p. Perhaps the most general method for solving the Dirichlet problem for these equations is the Perron method introduced independently by Perron [66] and Remak [67], and further refined in the linear case by Wiener and Brelot (and therefore often called the PWB method in the linear case). For the nonlinear case see Heinonen–Kilpeläinen–Martio [34] and the notes therein, and Björn–Björn–Shanmugalingam [13].

The Dirichlet problem, as posed above with f defined on the topological boundary $\partial\Omega$, is in many cases unnecessarily restrictive. For example, in the slit disk (see Example 5.2) one boundary value is prescribed for each point in the slit, even though it would be more natural to have two boundary values at those points (except for the tip), obtained by approaching the slit from either side. On the other hand, in some domains with complicated boundary there may be nontrivial parts of the boundary which are essentially invisible for the solutions and therefore should be treated accordingly in the Dirichlet problem.

For linear operators such as Δ, this drawback has been earlier addressed on $\mathbb{R}^n$ and Riemannian manifolds using the Martin boundary, see Martin [54], Ancona [4,5] and Anderson–Schoen [7]. The minimal Martin kernel functions, which compose the Martin boundary of the domain, are analogs of Poisson kernels for more irregular domains and provide us with integral representations for the solutions of the corresponding Dirichlet problem. In the slit disk one can see that there are two distinct minimal Martin kernels corresponding to each point in the slit (except for the tip). Although, as shown by e.g. Holopainen–Shanmugalingam–Tyson [41] and Lewis–Nyström [51], a Martin boundary can be defined even for nonlinear operators such as the p-Laplacian and its generalizations to metric spaces, we cannot hope to use the Martin boundary as a kernel for solving the corresponding Dirichlet problem in the nonlinear case.

The goal of this paper is to instead develop an alternative notion of boundary, called the prime end boundary, which can give rise to a more comprehensive potential theory suitable for the Perron method and taking the above geometrical concern into account. Prime ends were introduced by Carathéodory [20] in 1913 for simply connected planar domains. His approach is