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Circle and divisor problems, and double series of Bessel functions

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Abstract

In approximately 1915, Ramanujan recorded two identities involving doubly infinite series of Bessel functions. The identities were brought to the mathematical public for the first time when his lost notebook was published in 1988, and are connected with the classical, long-standing circle and divisor problems, respectively. We provide a proof of the first identity for the first time by analytically continuing a new kind of Dirichlet series. Delicate estimates of exponential sums are needed, and the new methods we introduce may be of independent interest.

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1. Introduction

On a page published with his lost notebook [13, p. 335], Ramanujan recorded two identities involving doubly infinite series of Bessel functions. One of them is connected with the classical *Dirichlet divisor problem*. The other, which is the focus of our paper, is associated with the

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equally famous *circle problem*. In order to state this identity, let $J_1(z)$ denote the ordinary Bessel function of order 1 [17, p. 15, Eq. (1)], and define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \dfrac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{1.1}$$

where $[x]$ is the integer part of x.

Entry 1.1. *If* $0 < \theta < 1$, $x > 0$, *and* $F(x)$ *is defined by* (1.1), *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x\left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}. \tag{1.2}$$

Note that the sum on the left side is finite. The double series in (1.2) does not converge absolutely, and it is not clear whether it is conditionally convergent or not. If the order of summation is *reversed* from that given by Ramanujan in Entry 1.1, then, as proved in [4], the double series is convergent and equality holds in (1.2). That proof crucially depends on the order of summation, and it is doubtful that it can be adapted to prove Entry 1.1. Also, in [2], extensive numerical calculations were effected on the double series of the second identity, in which the terms asymptotically have the same shape as those in (1.2), and the calculations did not provide convincing evidence that the double series is actually convergent.

When χ is a primitive character, the authors of [4] further derived an identity involving weighted divisor sums

$$d_\chi(n) = \sum_{k|n} \chi(k), \tag{1.3}$$

and Dirichlet L-functions. As a corollary, they proved that

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m\left(n+\frac{1}{4}\right)x}\right)}{\sqrt{m\left(n+\frac{1}{4}\right)}} - \frac{J_1\left(4\pi\sqrt{m\left(n+\frac{3}{4}\right)x}\right)}{\sqrt{m\left(n+\frac{3}{4}\right)}}\right\}, \tag{1.4}$$

where $r_2(n)$ denotes the number of representations of the positive integer n as a sum of two squares, where representations with different orders of the summands or different signs of the summands are regarded as distinct. Here, the prime $'$ on the summation sign on the left-hand side indicates that if x is a positive integer, then only $\frac{1}{2}r_2(x)$ is counted.

Identifying each representation of n as a sum of two squares with a lattice point within a circle of radius $\sqrt{x}$, Gauss observed that $\sum_{0\le n\le x} r_2(n)$ can be approximated by πx with an error term