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The smallest regular polytopes of given rank

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Abstract

An abstract polytope is called *regular* if its automorphism group has a single orbit on flags (maximal chains). In this paper, the regular n-polytopes with the smallest number of flags are found, for every rank $n > 1$. With a few small exceptions, the smallest regular n-polytopes come from a family of 'tight' polytopes with $2 \cdot 4^{n-1}$ flags, one for each n, with Schläfli symbol $\{4 \mid 4 \mid \cdots \mid 4\}$. Also with few exceptions, these have both the smallest number of elements, and the smallest number of edges in their Hasse diagram.

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1. Introduction

Abstract polytopes generalize the classical notion of convex geometric polytopes to more general structures. They are also known as thin residually-connected geometries. Highly symmetric examples include not only classical regular polytopes such as the Platonic solids and more exotic structures such as the 120-cell and 600-cell, but also non-degenerate regular maps on surfaces (such as Klein's quartic, of genus 3).

Roughly speaking, an abstract polytope $\mathcal{P}$ is a partially-ordered set endowed with a rank function, satisfying certain conditions that arise naturally from a geometric setting. Such objects were proposed by Grünbaum in the 1970s, and their definition (initially as 'incidence polytopes')

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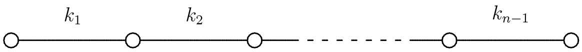

Fig. 1. Dynkin diagram for the Coxeter group $[k_1, k_2, \ldots, k_{n-1}]$.

and theory were developed by Danzer and Schulte, and described comprehensively in a book on the subject by McMullen and Schulte [11].

The elements of smallest rank (the atoms) are the vertices, those of the next rank are the edges, and so on, while those of maximal rank (the maximal elements) are the facets. The maximal chains are called *flags*. An automorphism (or symmetry) of $\mathcal{P}$ is any incidence- and rank-preserving bijection from $\mathcal{P}$ to $\mathcal{P}$. On the other hand, an incidence-reversing bijection from $\mathcal{P}$ to $\mathcal{P}$ is a *duality*. Every automorphism is uniquely determined by its effect on the flags of $\mathcal{P}$, and so the 'most symmetric' examples are those for which the automorphism group $\mathrm{Aut}(\mathcal{P})$ acts transitively, and hence regularly, on flags. These are *regular polytopes*.

The automorphism group of every regular polytope $\mathcal{P}$ of rank n is a quotient of some string Coxeter group $[k_1, k_2, \ldots, k_{n-1}]$, which is the universal group with Dynkin diagram given in Fig. 1.

Conversely, to be the automorphism group of a regular n-polytope, the quotient must be 'smooth' (in the sense that the orders of the generators and the products of pairs of the generators are preserved), and must satisfy a property known as the *intersection condition* (which ensures that the geometrically derived conditions hold).

The integers $k_1, k_2, \ldots, k_{n-1}$ have several interpretations (algebraic, combinatorial and geometric) and constitute what is known as the *type* of the regular polytope, denoted here by the so-called *Schläfli symbol* $\{k_1 \mid k_2 \mid \cdots \mid k_{n-1}\}$. It is normally assumed that each k_i is at least 3, for otherwise the Coxeter group is a direct product of two smaller Coxeter groups, and the polytope is a kind of direct sum of polytopes of smaller ranks, and not so interesting. (In particular, a regular polytope of type $\{2 \mid 2 \mid \cdots \mid 2\}$ has exactly two elements of each intermediate rank, with all possible incidences.)

A regular polytope of rank 2 and type $\{k\}$ is simply a regular k-gon, with k vertices and k edges, and $2k$ flags (given by the incident vertex–edge pairs), and dihedral automorphism group (of order $2k$). The Platonic solids and non-degenerate regular maps on orientable surfaces (of arbitrary genus) give nice examples of regular polytopes of rank 3, with type $\{k, m\}$ where k is the valency of each vertex and m is the size of each face.

For higher ranks, there are many families of regular polytopes, including those in the table below:

Name	Type	Comments
Regular n-simplex, $n \geq 2$	$\{3 \mid \overset{n-1}{\cdots} \mid 3\}$	Automorphism group S_{n+1}
Cross polytope (or n-orthoplex)	$\{3 \mid \overset{n-2}{\cdots} \mid 3 \mid 4\}$	Automorphism group $S_2 \wr S_n$
Hypercubic honeycombs	$\{4 \mid 3 \mid \overset{n-2}{\cdots} \mid 3 \mid 4\}$	Various examples

A related class of highly symmetric abstract polytopes are the so-called *chiral polytopes*, for which the automorphism group has exactly two orbits on flags, with the property that each flag Φ lies in a different orbit from its 'immediate neighbours' (the flags that differ from Φ in just one element). These are sometimes said to be maximally symmetric by rotation but not reflection—motivated by the analogous term 'chiral' for maps on surfaces. Their automorphism groups are smooth quotients of the orientation-preserving subgroup of index 2 in the appropriate string Coxeter group.