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Exceptional collections of line bundles on projective homogeneous varieties

Alexey Ananyevskiy[a], Asher Auel[b,*], Skip Garibaldi[b], Kirill Zainoulline[c]

[a] Department of Mathematics and Mechanics, St. Petersburg State University, Universitetsky Prospekt 28, St. Petersburg, 198504, Russia

[b] Department of Mathematics & Computer Science, Emory University, Atlanta, GA 30307, USA

[c] Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa ON K1N6N5, Canada

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Abstract

We construct new examples of exceptional collections of line bundles on the variety of Borel subgroups of a split semisimple linear algebraic group G of rank 2 over a field. We exhibit exceptional collections of the expected length for types A_2 and $B_2 = C_2$ and prove that no such collection exists for type G_2. This settles the question of the existence of full exceptional collections of line bundles on projective homogeneous G-varieties for split linear algebraic groups G of rank at most 2.

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0. Introduction

The existence question for full exceptional collections in the bounded derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(X)$ of a smooth projective variety X goes back to the foundational results of Beĭlinson [2,3] and Bernšteĭn–Gelfand–Gelfand [4] for $X = \mathbb{P}^n$. The works of Kapranov [18–21] suggested that the structure of projective homogeneous variety on X should imply the existence of full exceptional collections.

* Corresponding author.
E-mail addresses: alseang@gmail.com (A. Ananyevskiy), auel@mathcs.emory.edu, asher.auel@gmail.com (A. Auel), skip@mathcs.emory.edu (S. Garibaldi), kirill@uottawa.ca (K. Zainoulline).

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Conjecture. *Let X be a projective homogeneous variety of a split semisimple linear algebraic group G over a field of characteristic zero. Then there exists a full exceptional collection of vector bundles in $\mathsf{D}^{\mathrm{b}}(X)$.*

This conjecture remains largely unsolved, see [25, Section 1.1] for a recent survey of known results; for example, the conjecture is resolved for the Borel varieties of split groups of classical type and G_2. The bounded derived category $\mathsf{D}^{\mathrm{b}}(X)$ has come to be understood as a homological replacement for the variety X; exceptional collections provide a way to break up $\mathsf{D}^{\mathrm{b}}(X)$ into simple components. Such decompositions of the derived category can be seen as analogous to decompositions of the motive of X, a relationship that has been put into a conjectural framework by Orlov [28]. As an example, the existence of a full exceptional collection implies a splitting of the Chow motive $M(X)_{\mathbb{Q}}$ into tensor powers of Lefschetz motives (see [26]); this motivic decomposition is already known for projective homogeneous varieties of split linear algebraic groups (see [22]).

Let X be a variety over a field k. An object E of $\mathsf{D}^{\mathrm{b}}(X)$ is called *exceptional* if $\mathrm{Ext}^*(E, E) = k$; cf. [12, Def. 1.1], [6, Section 2]. Let W be a finite set and $\mathcal{P}$ a partial order on W. An ordered set (with respect to $\mathcal{P}$) of exceptional objects $\{E_w\}_{w\in W}$ in $\mathsf{D}^{\mathrm{b}}(X)$ is called a *$\mathcal{P}$-exceptional collection* if

$$\mathrm{Ext}^*(E_w, E_{w'}) = 0 \quad \text{for all } w <_{\mathcal{P}} w'.$$

If $\mathcal{P}$ is a total order, then a $\mathcal{P}$-exceptional collection is simply called an *exceptional collection*. A $\mathcal{P}$-exceptional collection $\{E_w\}_{w\in W}$ is called *full* if the smallest triangulated category containing $\{E_w\}_{w\in W}$ is $\mathsf{D}^{\mathrm{b}}(X)$ itself. Finally, a $\mathcal{P}$-exceptional collection of vector bundles $\{E_w\}_{w\in W}$ is said to be of the *expected length* if the classes $\{[E_w]\}_{w\in W}$ form a generating set of $K_0(X)$ of minimal cardinality. If $K_0(X)$ is a free abelian group (which is the case for all projective homogeneous varieties), then an exceptional collection is of the expected length if and only if its cardinality is the rank of $K_0(X)$. Note that any full $\mathcal{P}$-exceptional collection of vector bundles is of the expected length.

In the present paper we address the following closely related question.

Question. *Let X be the variety of Borel subgroups of a split semisimple linear algebraic group G and fix a partial order $\mathcal{P}$ on the Weyl group W of G. Does $\mathsf{D}^{\mathrm{b}}(X)$ have a $\mathcal{P}$-exceptional collection of the expected length consisting of line bundles?*

On the one hand, the question strengthens the conjecture by requiring the collection to consist of line bundles. On the other hand, it weakens the conjecture by allowing partial orders (such as the weak or strong Bruhat orders) instead of a total order and allowing the collection to merely generate $K_0(X)$. Partially ordered exceptional collections of vector bundles are considered in [5]; also see [17] for an alternate approach using Frobenius splitting in finite characteristic.

So far, a natural way to propagate known exceptional collections of line bundles is to use the result of Orlov [27, Cor. 2.7], that $\mathsf{D}^{\mathrm{b}}(X)$ has a full exceptional collection of line bundles if there exists a (Zariski locally trivial) projective bundle $X \to Y$ such that $\mathsf{D}^{\mathrm{b}}(Y)$ has a full exceptional collection of line bundles. More generally, $\mathsf{D}^{\mathrm{b}}(X)$ has a full exceptional collection of line bundles if X is the total space of a smooth Zariski locally trivial fibration, whose fiber and base both have derived categories with full exceptional collections of line bundles (see [9]). We remark that the result of Orlov on semiorthogonal decompositions of projective bundles (hence that of Beĭlinson and Bernšteĭn–Gelfand–Gelfand on $\mathbb{P}^n$) holds over an arbitrary noetherian base scheme; see