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Non-commutative functions and the non-commutative free Lévy–Hinčin formula

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Abstract

The paper is discussing infinite divisibility in the setting of operator-valued boolean, free and, more general, c-free independences. Particularly, using Hilbert bimodule and non-commutative function techniques, we obtain analogues of the Lévy–Hinčin integral representation for infinitely divisible real measures.

Keywords: Free probability; R-transform; Non-commutative functions; Fully matricial functions; Infinite divisibility

1. Introduction and notations

The paper presents some results concerning infinite divisibility in the framework of operator-valued non-commutative probability.

In probability theory – classic and non-commutative – limit theorems play a central role. The "most general" limit theorems involve so-called infinitesimal arrays, and the limit distributions are usually identified with *infinitely divisible* distributions. There is a consistent literature about

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infinitely divisible measures in classical probability (see [9]), dating back to Kolmogorov [10], P. Lévy [14] etc. Similar results have been found for non-commutative independences, such as [3,4] for free independence, [24] for boolean probability and [12,31] for conditionally free probability. In the operator-valued case, when states are replaced by positive conditional expectations or, more general, by completely positive maps between C^*- or operator algebras, very little was known about operator-valued infinite divisibility; the only exception we know of was Speicher's work [22]. One of the obstructions is that while in the scalar case important results characterizing infinite divisibility are coming from Nevanlinna–Pick representation properties of the functions that linearize additive convolutions (such as the log of the Fourier transform in the classic case or the Voiculescu's R- and ϕ-transforms in the free case), such analytic tools are not yet available in the operator-valued case. The new topic of non-commutative [11] or completely matricial [29] functions may possibly fill this gap. In particular, the results from Section 5 of the present paper indicate that results similar to the Nevanlinna–Pick representation hold also for non-commutative functions.

Besides the Introduction, the paper is organized in five sections. Section 2 presents some notations and results concerning certain maps on bimodules over a C^*-algebra. Sections 3–5 are aimed towards results characterizing infinite divisibility in operator-valued non-commutative probability using combinatorial and operator-algebra methods and constructions, in the spirit of [17,20]. More precisely, in Section 3 we use a non-commutative version of the "Boolean Fock space" construction from [1] to prove that, as in the scalar case, boolean infinite divisibility is trivial in the operator-valued case; particularly, any completely positive map between two C^*-algebras is boolean infinitely divisible. Section 4 is describing infinite divisibility with respect to free independence over a positive conditional expectation in terms of maps satisfying a condition of complete positivity. Section 5 is utilizing the techniques from the Boolean case (Section 3) to extend the results of Section 4 from positive conditional expectations to completely positive maps. In particular, we present a construction of the non-commutative version of the conditionally free R-transform of Bozejko, Leinert and Speicher in terms of creation, annihilation and preservation operators on certain inner-product bimodules. In the scalar-valued case this construction gives a new, combinatorial proof of the main result from [12] (see also [31]) characterizing conditionally free infinite divisibility. In Section 6 we use the tools from the theory of non-commutative functions (see [11,29]) to define the non-commutative R- (also constructed in [29]), ${}^cR$- and B-transforms. Reformulated in terms of these transforms, the results from Sections 3–5 are very similar to the free and conditionally free versions of the Lévy–Hinčin formula from [3], respectively [12] and [31]. The present material is using the notions detailed in [11], but it is self-contained in this regard, the needed material on non-commutative functions is briefly discussed in Section 6.2.

2. Notations and preliminary results on modules over a C^*-algebra

Throughout the paper $\mathcal{B}$ will be a unital C^*-algebra. We will denote by $\mathcal{B}\langle\mathcal{X}\rangle$ the $*$-algebra freely generated by $\mathcal{B}$ and the selfadjoint symbol $\mathcal{X}$. Unless otherwise explicitly stated, we do not suppose that $\mathcal{B}$ commutes with $\mathcal{X}$. We will also use the notations $\mathcal{B}\langle\mathcal{X}\rangle_0$ for the $*$-subalgebra of $\mathcal{B}\langle\mathcal{X}\rangle$ of all polynomials without a free term, and the notation $\mathcal{B}\langle\mathcal{X}_1, \mathcal{X}_2, \ldots\rangle$ for the $*$-algebra freely generated by $\mathcal{B}$ and the non-commutating selfadjoint symbols $\mathcal{X}_1, \mathcal{X}_2, \ldots$.

In several instances we will identify $\mathcal{T}(\mathcal{B})$, the tensor algebra over $\mathcal{B}$, to the subalgebra of $\mathcal{B}\langle\mathcal{X}\rangle$ spanned by $\{\mathcal{X}b_1\mathcal{X}b_2\cdots\mathcal{X}b_n : n \in \mathbb{N}, b_1, \ldots, b_n \in \mathcal{B}\}$ via

$$b_1 \otimes b_2 \otimes \cdots \otimes b_n \mapsto \mathcal{X}b_1\mathcal{X}b_2\cdots\mathcal{X}b_n.$$