Available online at www.sciencedirect.com
SciVerse ScienceDirect
ADVANCES IN Mathematics
Advances in Mathematics 237 (2013) 1–32
www.elsevier.com/locate/aim

On the full calculus of pseudo-differential operators on boundary groupoids with polynomial growth

Bing Kwan So

University of Luxembourg, Campus Kirchberg, Mathematics Research Unit, BLG 6, rue Richard Coudenhove-Kalergi, L-1359, Luxembourg

Received 8 December 2011; accepted 12 January 2013
Available online 10 February 2013

Communicated by Tomasz Mrowka

Abstract

In this paper, we enlarge the space of uniformly supported pseudo-differential operators on some groupoids by considering kernels satisfying certain asymptotic estimates. We show that such enlarged space contains the compact parametrix, and the generalized inverse of uniformly supported operators with Fredholm vector representation.

MSC: 58H05; 58J05; 35S05

Keywords: Groupoid; Singular pseudo-differential operator

1. Introduction

In this paper, we estimate the kernel of the generalized inverse of Fredholm elliptic operators defined on boundary groupoids.

Our work is motivated by [22], which is in turn motivated by the study of differential operators on manifolds with boundary [16,15].

Recall that in the classical construction, one first fixes a boundary defining function ρ, a smooth non-negative function on M with non-zero derivative on the boundary ∂M. Then an open neighborhood of $\partial M \subset M$ is identified with $[0, 1) \times \partial M$ (with $[0, 1)$ parameterized by ρ).

E-mail addresses: bkso@graduate.hku.hk, bingkwan.so@uni.lu.

doi:10.1016/j.aim.2013.01.001

Differential operators tangential to the boundary are written in the form $\rho\partial_\rho + \cdots$, and can be identified with kernels on the blowup M_b, known as the b-stretched product. The b-stretched product has three boundary defining functions ρ_{01}, ρ_{10} and ρ_{11}. By some explicit calculations, it can be shown that the generalized inverse of a Fredholm elliptic operator is a kernel with asymptotic expansion in ρ_{01}, ρ_{10}, ρ_{11}. The space of such kernels is known as the full calculus.

Following the classical theory, many variations emerge. Most notable is the work of Gil, Krainer and Mendoza. They considered 'cone operators' of the form $\rho^{-m}\Psi$, where Ψ is a b-differential operator as described above, using somewhat similar techniques (see [5–8], however the theory of cone operators mainly concerns geodesically incomplete spaces and is therefore beyond our scope).

Closer to our discussion, Lauter, Nistor and Monthubert studied the cusp, or c_n-calculus [10]. They begin to use some elements of pseudo-differential operators on a groupoid, which was first introduced by Nistor, Weinstein and Xu [19], and further developed by Ammann, Lauter and Nistor into so called Lie manifolds, or manifolds with Lie structure at infinity [2,11]. They prove that the Green function of elliptic operators has kernel that decays as a Schwartz function. However, these examples are quite similar to the manifold with the boundary case, and their argument makes use of the explicit structure of the underlying groupoid, described through the boundary defining function.

Ammann, et al. also apply similar theories to the example of polyhedral domains [1,3,20]. In particular, [20] considers the inverse of the differential operator as an element in the abstract C^*-algebra of the underlying groupoid.

The common theme of these results is that the Green function of elliptic differential operators is in general not compactly support supported kernels. One has to enlarge the calculus by considering non-compactly supported kernels of order $-\infty$, possibly non-smooth ones. Then one shows that the Green function lies in the enlarged calculus.

In the example of, say, natural differential operators on Poisson manifolds, however, there is no obvious notion of boundary defining functions. One can only use the theory of groupoid (pseudo)-differential operators to characterize these natural operators.

Motivated by the new class of examples, in [22], the author takes a more geometric approach. The groupoid is taken as the fundamental object, and one attempts to do computations without explicitly referring the singular structure. The idea was applied to the example of the symplectic groupoid of the Bruhat sphere, where it was shown that the parametrix of an elliptic, uniformed supported pseudo-differential operator is given by a groupoid pseudo-differential with exponentially decaying kernel.

Our main objective is to generalize the result of [22] to other similar groupoids, and also describe the generalized inverse of Fredholm operators. As far as we know, this paper is the first systematic study on non-uniformed supported groupoid pseudo-differential operators in some generality, besides the purely abstract C^*-algebra construction in [11,20]. Moreover, our work should clarify the role of the boundary defining function in these works, as well as the classical construction.

1.1. An overview of our approach

While the technical details are tedious and elementary, the idea behind our construction is actually very simple.

In Section 2, we recall some basic notions of pseudo-differential operators on a groupoid as in [19]. Then we define the notion of boundary groupoids. Essentially these groupoids are just