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Hamilton decompositions of regular expanders: A proof of Kelly's conjecture for large tournaments

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Abstract

A long-standing conjecture of Kelly states that every regular tournament on n vertices can be decomposed into $(n-1)/2$ edge-disjoint Hamilton cycles. We prove this conjecture for large n. In fact, we prove a far more general result, based on our recent concept of robust expansion and a new method for decomposing graphs. We show that every sufficiently large regular digraph G on n vertices whose degree is linear in n and which is a robust outexpander has a decomposition into edge-disjoint Hamilton cycles. This enables us to obtain numerous further results, e.g. as a special case we confirm a conjecture of Erdős on packing Hamilton cycles in random tournaments. As corollaries to the main result, we also obtain several results on packing Hamilton cycles in undirected graphs, giving e.g. the best known result on a conjecture of Nash-Williams. We also apply our result to solve a problem on the domination ratio of the Asymmetric Travelling Salesman problem, which was raised e.g. by Glover and Punnen as well as Alon, Gutin and Krivelevich.

Keywords: Tournaments; Hamilton decomposition; Robust expanders; Travelling salesman problem; Hamilton cycles

1. Introduction

1.1. Kelly's conjecture

A graph or digraph G has a Hamilton decomposition if it contains a set of edge-disjoint Hamilton cycles which together cover all the edges of G. The study of Hamilton decompositions is one of the oldest and most natural problems in Graph Theory. For instance, in 1892 Walecki

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showed that the complete graph K_n on n vertices has a Hamilton decomposition if n is odd (see e.g. [4,5,39]). Tillson [47] solved the corresponding problem for complete digraphs. Here every pair of vertices is joined by an edge in each direction, and there is a Hamilton decomposition unless the number of vertices is 4 or 6.

However, though there are several deep conjectures in the area, little progress has been made so far in proving results on Hamilton decompositions for general classes of graphs. Possibly the most well known problem in this direction is Kelly's conjecture from 1968 (see e.g. the monographs and surveys [7,10,36,40]), which states that every regular tournament has a Hamilton decomposition. Here a tournament is an orientation of a complete (undirected) graph. It is regular if the indegree of every vertex equals its outdegree. This condition is clearly necessary for a Hamilton decomposition. Here, we prove this conjecture for all large tournaments. In fact, it turns out that we can prove a much stronger result — we can obtain a Hamilton decomposition of any regular orientation of a sufficiently dense graph. More precisely, an oriented graph G is obtained by orienting the edges of an undirected graph. So it contains no cycles of length two (whereas in a digraph this is permitted).

Theorem 1.1. *For every $\varepsilon > 0$ there exists n_0 such that every r-regular oriented graph G on $n \geq n_0$ vertices with $r \geq 3n/8 + \varepsilon n$ has a Hamilton decomposition. In particular, there exists n_0 such that every regular tournament on $n \geq n_0$ vertices has a Hamilton decomposition.*

It is not clear whether the lower bound on r in Theorem 1.1 is best possible. However, as discussed below, there are oriented graphs whose in- and outdegrees are all very close to $3n/8$ but which do not contain even a single Hamilton cycle. Moreover, for $r < (3n-4)/8$, it is not even known whether an r-regular oriented graph contains a single Hamilton cycle (this is related to a conjecture of Jackson, see the survey [36] for a more detailed discussion). Both these facts indicate that any improvement in the lower bound on r would be extremely difficult to obtain.

Regular tournaments obviously exist only if n is odd, but we still obtain an interesting corollary in the even case. Suppose that G is a tournament on n vertices where n is even and which is as regular as possible, i.e. the in- and outdegrees differ by 1. Then Theorem 1.1 implies that G has a decomposition into edge-disjoint Hamilton paths. Indeed, add an extra vertex to G which sends an edge to all vertices of G whose indegree is below $(n-1)/2$ and which receives an edge from all others. The resulting tournament G' is regular, and a Hamilton decomposition of G' clearly corresponds to a decomposition of G into Hamilton paths.

The difficulty of Kelly's conjecture is illustrated by the fact that even the existence of two edge-disjoint Hamilton cycles in a regular tournament is not obvious. The first result in this direction was proved by Jackson [22], who showed that every regular tournament on at least 5 vertices contains a Hamilton cycle and a Hamilton path which are edge-disjoint. Zhang [48] then demonstrated the existence of two edge-disjoint Hamilton cycles. These results were improved by considering Hamilton cycles in oriented graphs of large in- and outdegree by Thomassen [46], Häggkvist [20], Häggkvist and Thomason [21] as well as Kelly, Kühn and Osthus [25]. Keevash, Kühn and Osthus [24] then showed that every sufficiently large oriented graph G on n vertices whose in- and outdegrees are all at least $(3n-4)/8$ contains a Hamilton cycle. This bound on the degrees is best possible and confirmed a conjecture of Häggkvist [20] (as mentioned above, there are extremal constructions which are almost regular). Note that this result implies that every sufficiently large regular tournament on n vertices contains at least $n/8$ edge-disjoint Hamilton cycles, whereas Kelly's conjecture requires $(n-1)/2$ edge-disjoint Hamilton cycles. The conjecture has also been proved for small values of n and for several special classes of tournaments (see e.g. [5,8] for somewhat outdated surveys).