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Highly oscillating thin obstacles

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Abstract

The focus of this paper is on a thin obstacle problem where the obstacle is defined on the intersection between a hyper-plane Γ in $\mathbb{R}^n$ and a periodic perforation $\mathcal{T}_\varepsilon$ of $\mathbb{R}^n$, depending on a small parameter $\varepsilon > 0$. As $\varepsilon \to 0$, it is crucial to estimate the frequency of intersections and to determine this number locally. This is done using strong tools from uniform distribution. By employing classical estimates for the discrepancy of sequences of type $\{k\alpha\}_{k=1}^{\infty}$, $\alpha \in \mathbb{R}$, we are able to extract rather precise information about the set $\Gamma \cap \mathcal{T}_\varepsilon$. As $\varepsilon \to 0$, we determine the limit u of the solution u_ε to the obstacle problem in the perforated domain, in terms of a limit equation it solves. We obtain the typical "strange term" behavior for the limit problem, but with a different constant taking into account the contribution of all different intersections, that we call the averaged capacity. Our result depends on the normal direction of the plane, but holds for a.e. normal on the unit sphere in $\mathbb{R}^n$.

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1. Introduction

1.1. Formulation of the problem

We consider the thin obstacle problem in a class of perforated domains. For $\varepsilon > 0$ we construct a perforated domain Γ_ε as follows. Let $Q_\varepsilon = (-\varepsilon/2, \varepsilon/2)^n$ and let $Q_\varepsilon(x) = x + (-\varepsilon/2, \varepsilon/2)^n$. Note that the cubes $Q_\varepsilon(\varepsilon k)$ for $k \in \mathbb{Z}^n$ are disjoint and

$$\bigcup_{k\in\mathbb{Z}^n} \overline{Q_\varepsilon(\varepsilon k)} = \mathbb{R}^n.$$

Next we perforate each cube by a small hole: let T be a compact subset of the unit ball B_1 with Lipschitz boundary ∂T, and for $a_\varepsilon < \varepsilon/2$ and $k \in \mathbb{Z}^n$, define $T_\varepsilon = a_\varepsilon T$ and $T_\varepsilon^k = a_\varepsilon T + \varepsilon k$. The set

$$\mathcal{T}_\varepsilon = \bigcup_{k\in\mathbb{Z}^n} T_\varepsilon^k$$

is to be thought of as a periodic background in the problem.

Let Ω be a domain in $\mathbb{R}^n$, and let $\Gamma = \Gamma_\nu$ be a hyper plane with surface measure σ, defined by

$$\Gamma_\nu = \{x \in \mathbb{R}^n : x \cdot \nu = x^0 \cdot \nu\} \tag{1}$$

for given $\nu \in S^{n-1}$ and $x^0 \in \mathbb{R}^n$.

The set

$$\Gamma_\varepsilon = \Gamma \cap \left(\bigcup_{k\in\mathbb{Z}^n} T_\varepsilon^k\right)$$

describes the intersection between the hyper-plane and the periodic background. Then, for a given $\psi \in L^\infty(\Omega) \cap H^1(\Omega)$ such that $\psi \le 0$ on $\partial\Omega$, we define the obstacle

$$\psi_\varepsilon = \psi\chi_{\Gamma_\varepsilon} = \begin{cases} \psi(x) & \text{if } x \in \Gamma_\varepsilon, \\ 0 & \text{if } x \notin \Gamma_\varepsilon, \end{cases}$$

and the admissible set

$$\mathcal{K}_{\psi_\varepsilon} = \{v \in H_0^1(\Omega) : v \ge \psi_\varepsilon\}. \tag{2}$$

The inequality in (2) is to be interpreted in the sense of trace, i.e. $\mathrm{Trace}_{\Gamma_\varepsilon}(u_\varepsilon - \psi) \ge 0$ on Γ_ε and $u_\varepsilon \ge 0$ a.e. in $\Omega \setminus \Gamma_\varepsilon$. We consider the following thin obstacle problem, for $f \in L^2(\Omega)$:

$$\begin{cases} \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon)dx \ge \int_\Omega (v - u_\varepsilon) f dx, & \text{for all } v \in \mathcal{K}_{\psi_\varepsilon}, \\ u_\varepsilon \in \mathcal{K}_{\psi_\varepsilon}. \end{cases} \tag{3}$$

The variational inequality (3) has a unique solution $u_\varepsilon \in \mathcal{K}_{\psi_\varepsilon}$ which can be obtained as the unique minimizer of the strictly convex and coercive functional

$$J(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 - f v dx, \quad v \in \mathcal{K}_{\psi_\varepsilon}.$$

We refer the reader to Evans [6] for the definition of trace and for the above minimization problem.