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# Admissibility of groups over function fields of p-adic curves

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**Abstract**

Let $K$ be a field and $G$ a finite group. The question of 'admissibility' of $G$ over $K$ was originally posed by Schacher, who gave partial results in the case $K = \mathbb{Q}$. In this paper, we give necessary conditions for admissibility of a finite group $G$ over function fields of curves over complete discretely valued fields. Using this criterion, we give an example of a finite group which is not admissible over $\mathbb{Q}_p(t)$. We also prove a certain Hasse principle for division algebras over such fields.


*Keywords:* Division algebras; Admissibility; Local-global principle; Patching; Galois groups; Function fields of curves; Complete discrete valued fields

## 0. Introduction

Let $K$ be a field and $G$ a finite group. We say that $G$ is *admissible* over $K$ if there exists a division ring $D$ central over $K$ and a maximal subfield $L$ of $D$ which is Galois over $K$ with Galois group $G$. Schacher asked, given a field $K$, which finite groups are admissible over $K$ and proved that if a finite group $G$ is admissible over $\mathbb{Q}$, then every Sylow subgroup of $G$ is metacyclic [21, 4.1]. This led to the conjecture that a finite group $G$ is admissible over $\mathbb{Q}$ if and only if every

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Sylow subgroup of $G$ is meta-cyclic. This conjecture has been proved for all solvable groups [24] and for certain non-solvable groups of small order [3,6–9].

Recently Harbater, Hartman and Krashen [14, 4.5] gave a characterization of admissible groups over function fields of curves over complete discretely valued fields with algebraically closed residue fields. In this paper, we consider the function fields of curves over complete discretely valued fields without any assumptions on the residue fields and prove the following

**Theorem 1.** *Let $K$ be a complete discretely valued field with residue field $k$ and $F$ be the function field of a curve over $K$. Let $G$ be a finite group of order $n$. Suppose $n$ is coprime to* $\mathrm{char}(k)$ *and $K$ contains a primitive $n$th root of unity. If $G$ is admissible over $F$ then every Sylow subgroup $P$ of $G$ has a normal series $P \supseteq P_1 \supseteq P_2$ such that*

(1) *$P/P_1$ and $P_2$ are cyclic*
(2) *$P_1/P_2$ is admissible over a finite extension of the residue field of a discrete valuation of $F$.*

A main ingredient in the proof of the above theorem is the following Hasse principle for central simple algebras, which has independent interest.

**Theorem 2.** *Let $K$ be a complete discretely valued field with residue field $k$ and $F$ be the function field of a curve over $K$. Let $A$ be a central simple algebra over $F$ of degree $n = \ell^r$ for some prime $\ell$ and $r \geq 1$. Assume that $\ell$ is not equal to* $\mathrm{char}(k)$ *and $K$ contains a primitive $n$th root of unity. Then* $\mathrm{index}(A) = \mathrm{index}(A \otimes F_v)$ *for some discrete valuation $v$ of $F$.*

For the proof of the above theorem, we use the patching techniques of [13]. A similar Hasse principle is proved for quadratic forms over such fields in [5, 3.1]. In [5, 4.3(ii)], it is proved that if a central simple algebra $A$ over $F$ ($F$ as above), is split over $F_v$ for all discrete valuations on $F$, then $A$ is split over $F$ (see also [15], 9.12).

There was no example, in the literature, of a finite group which is not admissible over a rational function field. Using Theorem 1, we give an example of a finite group which is not admissible over $\mathbb{Q}_p(t)$. We also prove admissibility of a certain class of groups over $\mathbb{Q}_p(t)$ using the patching techniques of [14].

**Theorem 3.** *Let $K$ be a $p$-adic field and $F$ the function field of a curve over $F$. Let $G$ be a finite group of order $n$. Suppose $n$ is coprime to $p$ and $K$ contains a primitive $n$th root of unity. If every Sylow subgroup of $G$ is a quotient of $\mathbb{Z}^4$, then $G$ is admissible over $F$.*

In [6], it was proved that every abelian group on three or less generators is admissible over $\mathbb{Q}(t)$. We prove that every abelian group of order $n$ with four or less generators is admissible over $\mathbb{Q}(\zeta)(t)$, where $\zeta$ is a primitive $n$th root of unity (4.6).

## 1. Some preliminaries

In this section we recall a few basic definitions and facts about division algebras and patching techniques [10,12,13,20–23].

Let $K$ be a field and $\mathrm{Br}(K)$ be the Brauer group of central simple algebras over $K$. For an integer $n \geq 2$, let ${}_n\mathrm{Br}(K)$ denote the $n$-torsion subgroup of $\mathrm{Br}(K)$. If $A$ and $B$ are two central simple algebras over $K$, we write $A \simeq B$ if $A$ and $B$ are isomorphic as $K$-algebras and we write $A = B$ if they represent the same element in $\mathrm{Br}(K)$. Let $n$ be an integer coprime to $\mathrm{char}(K)$. Suppose $E/K$ is a cyclic extension of degree $n$ and $\sigma$ a generator of $\mathrm{Gal}(E/K)$. For $b \in K^*$, let $(E/K, \sigma, b)$ (or simply $(E, \sigma, b)$) be the $K$-algebra generated by $E$ and $y$ with $y^n = b$ and