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# Inhomogeneous theory of dual Diophantine approximation on manifolds

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**Abstract**

The theory of inhomogeneous Diophantine approximation on manifolds is developed. In particular, the notion of nice manifolds is introduced and the divergence part of the Groshev type theory is established for all such manifolds. Our results naturally incorporate and generalize the homogeneous measure and dimension theorems for non-degenerate manifolds established to date. The results have natural applications beyond the standard inhomogeneous theory such as Diophantine approximation by algebraic integers.


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## 1. Introduction

### *1.1. Extremality, the Khintchine–Groshev theory and beyond*

Throughout $\mathbb{R}^+ = (0, +\infty)$, $|\cdot|$ denotes the supremum norm, $\|\cdot\|$ is the distance to the nearest integer and $\mathbf{a} \cdot \mathbf{b} := a_1 b_1 + \cdots + a_n b_n$ is the standard inner product of vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$.

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The point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *very well approximable* (abbr. *VWA*) if there exists $\varepsilon > 0$ such that

$$\|\mathbf{a} \cdot \mathbf{y}\| < |\mathbf{a}|^{-(1+\varepsilon)n} \tag{1}$$

holds for infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. By Dirichlet's theorem, when $\varepsilon = 0$ for all $\mathbf{y} \in \mathbb{R}^n$ inequality (1) holds for infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Thus, the essence of the definition of very well approximable points is that for these points the exponent within (1) can be improved beyond the trivial.

A relatively straightforward application of the Borel–Cantelli Lemma yields that almost every point $\mathbf{y} \in \mathbb{R}^n$ is not VWA. However, restricting $\mathbf{y}$ to a proper submanifold $\mathcal{M}$ of $\mathbb{R}^n$ introduces major difficulties in attempting to describe the measure theoretic structure of the VWA points $\mathbf{y} \in \mathcal{M}$. Essentially, it is this investigation that has given rise to the now flourishing area of 'Diophantine approximation on manifolds' within metric number theory.

Diophantine approximation on manifolds dates back to the 1930s with a conjecture of Mahler [50] in transcendence theory. Using the above terminology, the conjecture states that almost all points on the Veronese curve

$$\mathcal{V}_n := \{(x, \ldots, x^n) : x \in \mathbb{R}\}$$

are not VWA. Mahler's conjecture remained a key open problem in metric number theory for over thirty years and was eventually solved by Sprindžuk [55]. Moreover, its solution led Sprindžuk [56] to make an important general conjecture. He claimed that any analytic non-degenerate[1] submanifold of $\mathbb{R}^n$ satisfies a similar property which we now make precise. A differentiable manifold $\mathcal{M}$ in $\mathbb{R}^n$ is said to be *extremal* if almost all points of $\mathcal{M}$ (with respect to the natural Riemannian measure on $\mathcal{M}$) are not VWA.

Related, but far more delicate problems arise when, instead of (1), one considers the inequality

$$\|\mathbf{a} \cdot \mathbf{y}\| < \Pi_+(\mathbf{a})^{-1-\varepsilon}, \tag{2}$$

where

$$\Pi_+(\mathbf{a}) = \prod_{i=1}^{n} \max\{1, |a_i|\}.$$

The point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *very well multiplicatively approximable* (abbr. *VWMA*) if there exists $\varepsilon > 0$ such that (2) holds for infinitely many $\mathbf{a} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. A differentiable manifold $\mathcal{M}$ in $\mathbb{R}^n$ is said to be *strongly extremal* if almost all points of $\mathcal{M}$ are not VWMA. It is easily verified that any VWA point $\mathbf{y}$ is VWMA and so any strongly extremal manifold is extremal. Baker [3] suggested the far-reaching generalization of Mahler's problem that Veronese curves are strongly extremal. This was later extended to manifolds by Sprindžuk [56]:

BAKER–SPRINDŽUK CONJECTURE: *Any analytic non-degenerate submanifold of $\mathbb{R}^n$ is strongly extremal.*

This fundamental conjecture was proved in 1998 by Kleinbock and Margulis in their landmark paper [47] for arbitrary (not necessarily analytic) non-degenerate manifolds. Essentially, non-degenerate manifolds are smooth sub-manifolds of $\mathbb{R}^n$ which are sufficiently curved so as to deviate from any hyperplane. Formally, a manifold $\mathcal{M}$ of dimension $m$ embedded in $\mathbb{R}^n$ is said

[1] The notion of non-degeneracy will be formally introduced below.