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Scattering for wave maps exterior to a ball

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Abstract

We consider 1-equivariant wave maps from $\mathbb{R}_t \times (\mathbb{R}^3_x \setminus B) \to S^3$ where B is a ball centered at 0, and ∂B gets mapped to a fixed point on S^3. We show that 1-equivariant maps of degree zero scatter to zero irrespective of their energy. For positive degrees, we prove asymptotic stability of the unique harmonic maps in the energy class determined by the degree.

Keywords: Nonlinear wave equations; Sigma models; Concentration compactness; Dispersive theory

1. Introduction

Wave maps, also known as nonlinear σ-models, are a well-studied area in physics and mathematics. They constitute a class of nonlinear wave equations defined as critical points (at least formally) of Lagrangians

$$\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}^{d+1}} \frac{1}{2}\left(-|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2\right) dt dx$$

where $u : \mathbb{R}^{d+1} \to M$ is a smooth map into a Riemannian manifold (M, g). If $M \hookrightarrow \mathbb{R}^N$ is embedded, then critical points are characterized by the property that $\Box u \perp T_u M$ where $\Box$ is the d'Alembertian. In particular, harmonic maps from $\mathbb{R}^d \to M$ are wave maps which do not depend

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on time. For a recent review of some of the main developments in the area we refer to Krieger's survey [10].

In the presence of symmetries, such as when the target manifold M is rotationally symmetric, one often singles out a special class of such maps called equivariant wave maps. For example, for the sphere $M = S^d$ one requires that $u \circ \rho = \rho^\ell \circ u$ where ℓ is a positive integer and $\rho \in SO(d)$ acts on both $\mathbb{R}^d$ and S^d by rotation, in the latter case about a fixed axis. These maps themselves have been extensively studied, see for example Shatah [14], Christodoulou and Tahvildar-Zadeh [6], Shatah and Tahvildar-Zadeh [15]. For a summary of these developments, see the book Shatah and Struwe [16].

In this paper, we investigate equivariant wave maps from $3+1$-dimensional Minkowski space exterior to a ball and with S^3 as target. To be specific, let $B \subset \mathbb{R}^3$ be the unit ball in $\mathbb{R}^3$. We then consider wave maps $U : \mathbb{R} \times (\mathbb{R}^3 \setminus B) \to S^3$ with a Dirichlet condition on ∂B, i.e., $U(\partial B) = \{N\}$ where N is a fixed point on S^3. In the usual equivariant formulation of this equation, where ψ is the azimuth angle measured from the north pole, the equation for the ℓ-equivariant wave map from $\mathbb{R}^{3+1} \to S^3$ reduces to

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \ell(\ell+1)\frac{\sin(2\psi)}{2r^2} = 0. \tag{1}$$

We restrict to $\ell = 1$ and $r \geq 1$ with Dirichlet boundary condition $\psi(1,t) = 0$ for all $t \geq 0$. In other words, we are considering the Cauchy problem

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0, \quad r \geq 1, \\ &\psi(1,t) = 0, \quad \forall t \geq 0, \\ &\psi(r,0) = \psi_0(r), \\ &\psi_t(r,0) = \psi_1(r). \end{aligned} \tag{2}$$

The conserved energy is

$$\mathcal{E}(\psi, \psi_t) = \int_1^\infty \frac{1}{2}\left(\psi_t^2 + \psi_r^2 + 2\frac{\sin^2(\psi)}{r^2}\right) r^2\, dr. \tag{3}$$

Any $\psi(r,t)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(\infty, t) = n\pi$ for all $t \in I$ where $n \geq 0$ is fixed.

The natural space to place the solution into for $n = 0$ is the *energy space* $\mathcal{H} := (\dot{H}_0^1 \times L^2)((1,\infty))$ with norm

$$\|(\psi, \dot{\psi})\|_{\mathcal{H}}^2 := \int_1^\infty (\psi_r^2(r) + \dot{\psi}^2(r))\, r^2\, dr. \tag{4}$$

Here $\dot{H}_0^1((1,\infty))$ is the completion of the smooth functions on $(1,\infty)$ with compact support under the first norm on the right-hand side of (4).

The exterior equation (2) was proposed by Bizoń et al. [3] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps. In the physics literature, this model was introduced in [2] as an easier alternative to the Skyrmion equation. Moreover, [2] stresses the analogy with the damped pendulum which plays an important role in our analysis. Numerical simulations described in [3] indicate that in each equivariance class and topological class given by the boundary value $n\pi$ at $r = \infty$ *every solution* scatters