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Non-uniform painless decompositions for anisotropic Besov and Triebel–Lizorkin spaces

Carlos Cabrelli, Ursula Molter*, José Luis Romero

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Capital Federal, Argentina
IMAS, UBA-CONICET, Argentina

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Abstract

In this article we construct affine systems that provide a simultaneous atomic decomposition for a wide class of functional spaces including the Lebesgue spaces $L^p(\mathbb{R}^d)$, $1 < p < +\infty$. The novelty and difficulty of this construction is that we allow for non-lattice translations.

We prove that for an arbitrary expansive matrix A and any set Λ—satisfying a certain spreadness condition but otherwise irregular—there exists a smooth window whose translations along the elements of Λ and dilations by powers of A provide an atomic decomposition for the whole range of the anisotropic Triebel–Lizorkin spaces. The generating window can be either chosen to be bandlimited or to have compact support.

To derive these results we start with a known general "painless" construction that has recently appeared in the literature. We show that this construction extends to Besov and Triebel–Lizorkin spaces by providing adequate dual systems.

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Keywords: Besov spaces; Triebel–Lizorkin spaces; Anisotropic function spaces; Non-uniform atomic decomposition; Affine systems

* Corresponding author at: Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Capital Federal, Argentina.

E-mail addresses: cabrelli@dm.uba.ar (C. Cabrelli), umolter@dm.uba.ar (U. Molter), jlromero@dm.uba.ar (J.L. Romero).

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1. Introduction

The membership of a distribution to a large number of classical functional spaces can be characterized by its Littlewood–Paley decomposition, which consists of a sequence of smooth frequency cut-offs at dyadic scales. The functional spaces that can be described in that way are generically known as Besov and Triebel–Lizorkin spaces. This class includes among others the Lebesgue spaces $L^p(\mathbb{R}^d)$, $(1 < p < +\infty)$, Sobolev spaces and Lipschitz spaces.

More recently, anisotropic variants of these spaces have been introduced, where the dyadic scales are replaced by more general ones allowing different spatial directions to be dilated by different factors (see [31,35,24,38,8,11,9,10] and the references therein). These variants are useful for example to study anisotropic smoothness conditions.

Time-scale atomic decompositions are a very powerful tool to analyze Besov and Triebel–Lizorkin spaces. The technique consists of representing a general distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ as a superimposition of atoms $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$,

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k}.$$

The membership of f to a particular Besov or Triebel–Lizorkin space is characterized by the decay of its coefficients $c_{j,k}$. In the classical (isotropic) case, the atoms are of the form $\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k)$, where ψ is an adequate window function called a wavelet. In the anisotropic case a number of alternatives are possible. One of them is to replace the powers of 2 by powers of a more general matrix A, yielding atoms of the form,

$$\{\, |\det(A)|^{-j/2}\psi(A^{-j}\cdot -k) \mid j \in \mathbb{Z}, k \in \mathbb{Z}^d \,\}.$$

(See [38] for other alternatives.) The existence of atomic decompositions for anisotropic Besov and Triebel–Lizorkin spaces is a well-known fact. There is an ample literature giving sets of atoms with specific properties (see [38] and the references therein). The purpose of this article is to show that these spaces also admit atomic decompositions where the integer translations are replaced by translations along quite arbitrary sets. Given a matrix $A \in \mathbb{R}^{d\times d}$ that is *expansive* (i.e. all its eigenvalues μ satisfy $|\mu| > 1$) and a set $\Lambda \subseteq \mathbb{R}^d$ that satisfies a certain spreadness condition but is otherwise irregular, we show that there exists a function $\psi \in \mathcal{S}$ such that the irregular time-scale system,

$$W(\psi, A, \Lambda) := \{\, |\det(A)|^{-j/2}\psi(A^{-j}\cdot -\lambda) \mid j \in \mathbb{Z}, \lambda \in \Lambda \,\}, \tag{1}$$

gives an atomic decomposition of the whole class of (anisotropic) Besov and Triebel–Lizorkin spaces. The function ψ can be chosen to be bandlimited or to have compact support (in this latter case, since ψ cannot have infinitely many vanishing moments, we have to restrict the decomposition to a subclass of spaces having a bounded degree of smoothness).

This result is new even in the isotropic case. Its relevance stems from the fact that the set of translation nodes Λ is not assumed to have any kind of regularity (besides a mild spreadness condition). As a comparison to our work, the only results giving such irregular time-scale decompositions that we are aware of are: the one that comes from the general theory of atomic decompositions of coorbit spaces associated with group representations [19], and the one obtained from oversampling Calderón's continuous resolution of the identity [25,30]. These prove the existence of time-scale atomic decompositions using irregular sets of translates, *as long as they are sufficiently dense* (in a sense that may be hard to quantify). In contrast, our result proves that *any* set of translates can be used (under a mild spreadness assumption).