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Distributionally concave symmetric spaces and uniqueness of symmetric structure

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Abstract

We present new results concerning the uniqueness of symmetric structure of symmetric function spaces. Our methods are partly based on a detailed study of distributionally concave spaces and the tensor product operator.

Keywords: Symmetric space; Orlicz space; Orlicz–Lorentz space; Distributionally concave space; Uniqueness of symmetric structure; Tensor product operator

1. Introduction

To better explain the motivation and main results of this paper, we first cite the following important result from the Memoir of Johnson et al. [15], which extends a well-known result that every $L_p[0, 1]$-space, $1 \le p \le \infty$ has a unique representation as a symmetric space on $[0, 1]$ [24, 2.e.8].

Theorem 1.1 (*[15, Corollary 7.9]*)**.** *Let a separable Orlicz space $L_M = L_M[0, 1]$ be p-convex for some $p > 2$. Then the following conditions are equivalent.*

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(1) *If a symmetric function space X is isomorphic to a subspace of the space L_M, then either $X = L_2[0,1]$ or $X = L_M[0,1]$ (up to norm equivalence).*
(2) *For some $C > 0$ and any $u, v \geq 1$ the following holds*

$$M(uv) \leq CM(u)M(v). \tag{1.1}$$

In particular, the condition (1.1) provides the uniqueness of symmetric structure of a p-convex ($p > 2$) separable Orlicz space L_M (that is, an arbitrary symmetric space X on $[0, 1]$ which is isomorphic to L_M can be equivalently renormed so that $X = L_M$). Comparable results were also achieved for Lorentz spaces $\Lambda(p, \psi)$, $p > 2$ [13,14].

Our analysis of the proof of Theorem 1.1 in [15] and Carothers' results in [13,14] have led us to believe that they are all underpinned by a certain geometric property of symmetric function spaces, which we have termed "the distributional concavity" in this paper. In the present paper the result of Theorem 1.1 is extended to a much wider class of distributionally concave symmetric function spaces, which contains both Orlicz and Lorentz spaces (moreover, it contains the class of so-called Orlicz–Lorentz spaces). The distributional concavity was introduced by Montgomery-Smith and Semenov in [32][1] (see also an alternative approach to this property in [38] and discussion in the appendix to [28]). This property is a natural counterpart of the notion of distributional convexity introduced originally by Kalton in [16]. The class of distributionally concave symmetric function spaces appears to be of interest in its own right; however, the treatment of this class in [32] is rather patchy. In particular, the end of the proof of Theorem 22 in [32] (which provides an important characterization of this class) contains a mistake. In this paper, we rectify this mistake and provide the complete proof of that theorem together with a number of corollaries. Using this opportunity, we present in the last section a detailed exposition of distributionally concave symmetric function spaces and its properties. We thank Professors Montgomery-Smith and Semenov for numerous discussions concerning [32] and their encouragement to publish our account of their results.

The main results are Theorems 1.2 and 1.3 and Corollaries 1.4 and 1.5 below. Another important technical advantage of our approach to the isomorphic classification of symmetric spaces is the consistent usage of the tensor product operator (which is a well-known tool in interpolation theory). The latter operator lurks in the background of a number of proofs in [24,15,13,14]; however, its importance in the isomorphic theory of symmetric function spaces appears to have been overlooked. Indeed, in the class of symmetric spaces which is studied in this paper, the boundedness of this operator is, in fact, equivalent to the uniqueness of symmetric structure as evidenced from Corollary 1.4 below.

Theorem 1.2. *Let E be a distributionally concave separable symmetric function space on* $[0, 1]$, *such that*

(a) *E is an interpolation space for the couple (L_2, L_∞)*

and

(b) *the tensor product operator $(x, y) \to x \otimes y$ is bounded from the Cartesian square $E \times E$ into E.*

For every symmetric space F isomorphic to a subspace of E, one of the following options holds.

(1) *Either $F = L_2[0, 1]$ or $F = E$.*
(2) *The Haar system in F is equivalent to a sequence of disjointly supported functions in E.*

[1] Termed D^*-convexity there.