Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 232 (2013) 543–570

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Cantor boundary behavior of analytic functions

Xin-Han Dong[a,∗], Ka-Sing Lau[b,a], Jing-Cheng Liu[a,b]

[a] *Key Laboratory of High Performance Computing and Stochastic Information Processing (Ministry of Education of China), College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China*

[b] *Department of Mathematics, The Chinese University of Hong Kong, Hong Kong*

Received 7 June 2011; accepted 22 September 2012

Available online 23 October 2012

Communicated by Kenneth Falconer

Abstract

Let $A(\mathbb{D})$ be the space of analytic functions on the open disk $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$, we are interested in the class of $f \in A(\mathbb{D})$ such that the image $f(\partial\mathbb{D})$ is a curve that forms loops everywhere. This fractal behavior was first raised by Lund et al. (1998) [21] in the study of the Cauchy transform of the Hausdorff measure on the Sierpinski gasket. We formulate the property as the *Cantor boundary behavior* (CBB) and establish two sufficient conditions through the distribution of zeros of $f'(z)$ and the mean growth rate of $|f'(z)|$ near the boundary. For the specific cases we carry out a detailed investigation on the gap series and the complex Weierstrass functions; the CBB for the Cauchy transform on the Sierpinski gasket will appear elsewhere.

© 2012 Elsevier Inc. All rights reserved.

MSC: primary 30C 35; secondary 30C 55; 28A 80

Keywords: Analyticity; Boundary behavior; Blaschke product; Cantor set; Conformal; Fractal; Growth rate; Lacunary series; Simply connected; Univalence; Weierstrass function; Zeros

1. Introduction

Let $\mathbb{D}$ be the open unit disk and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For f analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, let $\partial f(\mathbb{D})$ denote the boundary of $f(\mathbb{D})$. It follows from the open mapping

∗ Corresponding author.

E-mail addresses: xhdong@hunnu.edu.cn (X.-H. Dong), kslau@math.cuhk.edu.hk (K.-S. Lau), liujingcheng11@126.com (J.-C. Liu).

0001-8708/$ - see front matter © 2012 Elsevier Inc. All rights reserved.

doi:10.1016/j.aim.2012.09.021

theorem that $\partial f(\mathbb{D}) \subset f(\partial\mathbb{D})$. These two sets have very rich and intriguing geometric properties. In fact, when f is conformal on $\mathbb{D}$, then they are equal and there is a large literature on their boundary behaviors; the reader can refer to Pommerenke [28] for the classical developments; there are more recent developments in connection with Brownian motion [20] and for the random fractals [3,4].

Our interest is in the class of analytic functions f for which the image curve $f(\partial\mathbb{D})$ form infinitely many loops everywhere; naturally they are not univalent. From intuition, the function f has the property that for any open arc I on $\partial\mathbb{D}$, $f(I)$ contains at least one loop (which is inside $f(\mathbb{D})$). If we let $\mathcal{C} = f^{-1}(\partial f(\mathbb{D}))$, then $\mathcal{C} = \partial\mathbb{D} \setminus \bigcup_{i=1}^{\infty} I_i$, where I_i are open arcs of $\partial\mathbb{D}$, $f(I_i) \subset f(\mathbb{D})$, and $\overline{\bigcup_{i=1}^{\infty} I_i} = \partial\mathbb{D}$. The condition of loops everywhere implies that $\mathcal{C}$ is a nowhere dense closed set (i.e., totally disconnected) and the image stretches out to be $f(\mathcal{C}) = \partial f(\mathbb{D})$, which is a curve if $f(\mathbb{D})$ is simply connected, or contains more than one curve if $f(\mathbb{D})$ is multiple connected. This is analogous to the Cantor function that maps the Cantor set onto the interval [0, 1]. Also note that for any uncountable nowhere dense closed set $E \subset \partial\mathbb{D}$, if we let $E' \subseteq E$ be the set of accumulation points x of E such that each neighborhood of x contains *uncountably* many points of E, then E' is, in addition, a perfect set (no isolated point), and it is well known that such E' is homeomorphic to the Cantor set [17, p. 100]. For this reason we call an uncountable nowhere dense closed subset $E \subset \partial\mathbb{D}$ a *Cantor-type set* for convenience.

This boundary behavior was first observed by Lund et al. [21] in the study of Cauchy transform on the Sierpinski gasket. Let μ be the canonical Hausdorff measure on the Sierpinski gasket K and let $F(z) = \int_K d\mu(w)/(z - w)$ be the Cauchy transform. It is clear that F is analytic outside K, and they showed that F has a Hölder continuous extension over K. Let Δ be the unbounded connected component of $\mathbb{C} \setminus K$. From computer graphics, they observed that the image $F(\partial\Delta)$ is a complicate system of loops. They raised the *Cantor set conjecture* that $F^{-1}(\partial F(\Delta))$ is a Cantor-type set (see also [7,8]). By using the Riemann mapping theorem, we can convert it into the more general problem on the unit disk $\mathbb{D}$ as the above.

To formulate such boundary behavior, there are difficulties in obtaining a precise meaning of "infinitely many loops". Our approach is to use a weaker topological concept on the connected components determined by $f(\partial\mathbb{D})$. We let $\widehat{\mathbb{C}}$ be the Riemann sphere. We make a first decomposition on the range as

$$\widehat{\mathbb{C}} \setminus f(\partial\mathbb{D}) = \bigcup_j \mathcal{W}_j \tag{1.1}$$

where $\mathcal{W}_j$ are connected components (see section 2); then a second decomposition on the domain by

$$f^{-1}(\mathcal{W}_j) = \bigcup_{k=1}^{q_j} O_j^k \tag{1.2}$$

with O_j^k connected components and $q_j < \infty$. It follows that these $\mathcal{W}_j$, O_j^k are simply connected and f satisfies $f(O_j^k) = \mathcal{W}_j$ and $f(\partial O_j^k) = \partial\mathcal{W}_j$ (Theorem 2.4), this is equivalent to say that $f : O_j^k \to \mathcal{W}_j$ is a *proper map*, in the sense that pre-images of compact subsets in the range are compact [29].

Definition 1.1. We say that $f \in A(\mathbb{D})$ has the *Cantor boundary behavior (CBB)* if (i) $f^{-1}(\partial f(\mathbb{D}))$ and (ii) $f^{-1}(\partial\mathcal{W}_j) \cap \partial\mathbb{D}$ for each j, are Cantor-type set in $\partial\mathbb{D}$.

Download English Version:

https://daneshyari.com/en/article/4666135

Download Persian Version:

https://daneshyari.com/article/4666135

Daneshyari.com