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The ternary commutator obstruction for internal crossed modules

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Abstract

In finitely cocomplete homological categories, co-smash products give rise to (possibly higher-order) commutators of subobjects. We use binary and ternary co-smash products and the associated commutators to give characterisations of internal crossed modules and internal categories, respectively. The ternary terms are redundant if the category has the *Smith is Huq* property, which means that two equivalence relations on a given object commute precisely when their normalisations do. In fact, we show that the difference between the Smith commutator of such relations and the Huq commutator of their normalisations is measured by a ternary commutator, so that the *Smith is Huq* property itself can be characterised by the relation between the latter two commutators. This allows us to show that the category of loops does not have the *Smith is Huq* property, which also implies that ternary commutators are generally not decomposable into nested binary ones.

Thus, in contexts where *Smith is Huq* need not hold, we obtain a new description of internal categories, Beck modules and double central extensions, as well as a decomposition formula for the Smith commutator. The ternary commutator now also appears in the Hopf formula for the third homology with coefficients in the abelianisation functor.

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0. Introduction

Internal crossed modules in a semi-abelian category [38] were introduced by Janelidze in [36]. His starting point is the desired correspondence between crossed modules and internal categories, which determines the basic properties that crossed modules should satisfy. His definition is based on the concept of internal action which he introduced with Bourn in [17] and which is further worked out in [9].

He explains that the extension of the case of groups to semi-abelian categories is not entirely without difficulties. The most straightforward description of the concept of crossed module merely gives so-called *star-multiplicative graphs*—in which the composition of morphisms is only defined locally around the origin—and not the internal groupoids one would expect, in which every composable pair of morphisms can actually be composed. This defect can be mended, as it is indeed done in [36]. Unfortunately, the resulting characterisation of internal crossed modules becomes slightly more complicated than expected after considering the groups case.

This gave rise to the question, whether every star-multiplicative graph can be equipped with a unique internal groupoid structure. It turns out [53] that the gap between the two is precisely as big as the gap between the Huq commutator of normal subobjects and the Smith commutator of internal equivalence relations. That is to say, in a semi-abelian category they are equivalent if and only if the *Smith is Huq* condition holds. This explains why the difference between the two concepts is invisible in the category of groups, in fact in any of the concrete algebraic categories where internal crossed modules were ever studied: all of those are strongly protomodular (and action accessible), which as we know implies the *Smith is Huq* condition.

Introducing ternary commutators gives a different view on the situation, more natural in a sense: just as internal groupoids can be described as internal reflexive graphs with a certain binary (Smith) commutator being trivial, now we can say that internal groupoids may also be described as internal star-multiplicative graphs for which a certain ternary (Higgins) commutator is zero. Equivalently, a certain coherence condition involving ternary co-smash products holds for the associated internal precrossed module (satisfying the Peiffer condition). A byproduct of this analysis is that the context is enlarged to a non-exact setting (being careful with the notion of star-multiplicativity), as we may mostly work in finitely cocomplete homological categories instead of semi-abelian ones.

0.1. Internal actions

It is well known that *every split epimorphism of groups is a semi-direct product projection*. This fact gives rise to an equivalence between the category $\mathsf{Pt}_B(\mathsf{Gp})$ of split epimorphisms of groups (with chosen splitting) with codomain B and the category of B-groups. Similarly [17,9], in a semi-abelian category $\mathcal{A}$, internal actions correspond to split epimorphisms. Furthermore, since the kernel functor $\mathsf{Pt}_B(\mathcal{A}) \to \mathcal{A}$ is monadic for every object B in $\mathcal{A}$, the internal actions are defined as the algebras over the corresponding monad.