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Invariant subalgebras of affine vertex algebras

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Dedicated to my father Michael A. Linshaw, M. D., on the occasion of his 70th birthday.

Abstract

Given a finite-dimensional complex Lie algebra $\mathfrak{g}$ equipped with a nondegenerate, symmetric, invariant bilinear form B, let $V_k(\mathfrak{g}, B)$ denote the universal affine vertex algebra associated to $\mathfrak{g}$ and B at level k. For any reductive group G of automorphisms of $V_k(\mathfrak{g}, B)$, we show that the invariant subalgebra $V_k(\mathfrak{g}, B)^G$ is strongly finitely generated for generic values of k. This implies the existence of a new family of deformable $\mathcal{W}$-algebras $\mathcal{W}(\mathfrak{g}, B, G)_k$ which exist for all but finitely many values of k.

Keywords: Invariant theory; Affine vertex algebra; Current algebra; Reductive group action; Orbifold construction; Strong finite generation; $\mathcal{W}$-algebra

1. Introduction

We call a vertex algebra $\mathcal{V}$ *strongly finitely generated* if there exists a finite set of generators such that the collection of iterated Wick products of the generators and their derivatives spans $\mathcal{V}$. Many known vertex algebras have this property, including affine, free field and lattice vertex algebras, as well as the $\mathcal{W}$-algebras $\mathcal{W}(\mathfrak{g}, f)_k$ associated via quantum Drinfeld–Sokolov reduction to a simple, finite-dimensional Lie algebra $\mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$. Strong finite generation has many important consequences, and in particular implies that both Zhu's associative algebra $A(\mathcal{V})$, and Zhu's commutative algebra $\mathcal{V}/C_2(\mathcal{V})$, are finitely generated.

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In recent work, we have investigated the strong finite generation of invariant vertex algebras $\mathcal{V}^G$, where G is a reductive group of automorphisms of $\mathcal{V}$. This is a vertex algebra analogue of Hilbert's theorem on the finite generation of classical invariant rings. It is a subtle and essentially "quantum" phenomenon that is generally destroyed by passing to the classical limit before taking invariants. Often, $\mathcal{V}$ admits a G-invariant filtration for which $\mathrm{gr}(\mathcal{V})$ is a commutative algebra with a derivation (i.e., an abelian vertex algebra), and the classical limit $\mathrm{gr}(\mathcal{V}^G)$ is isomorphic to $(\mathrm{gr}(\mathcal{V}))^G$ as a commutative algebra. Unlike $\mathcal{V}^G$, $\mathrm{gr}(\mathcal{V}^G)$ is generally not finitely generated as a vertex algebra, and a presentation will require both infinitely many generators and infinitely many relations.

Isolated examples of this phenomenon have been known for many years (see for example [3,6,5,8,13]), although the first general results of this kind were obtained in [18], in the case where $\mathcal{V}$ is the $\beta\gamma$-system $\mathcal{S}(V)$ associated to the vector space $V = \mathbb{C}^n$. The full automorphism group of $\mathcal{S}(V)$ preserving a natural conformal structure is GL_n. By a theorem of Kac–Radul [11], $\mathcal{S}(V)^{GL_n}$ is isomorphic to the vertex algebra $\mathcal{W}_{1+\infty}$ with central charge $-n$. In [19] we showed that $\mathcal{W}_{1+\infty,-n}$ has a minimal strong generating set consisting of $n^2 + 2n$ elements, and in particular is a $\mathcal{W}$-algebra of type $\mathcal{W}(1, 2, \ldots, n^2 + 2n)$. For an arbitrary reductive group $G \subset GL_n$, $\mathcal{S}(V)^G$ decomposes as a direct sum of irreducible, highest-weight $\mathcal{W}_{1+\infty,-n}$-modules. The strong finite generation of $\mathcal{W}_{1+\infty,-n}$ implies a certain finiteness property of the modules appearing in $\mathcal{S}(V)^G$. This property, together with a classical theorem of Weyl, yields the strong finite generation of $\mathcal{S}(V)^G$. Using the same approach, we also proved in [18] that invariant subalgebras of bc-systems and $bc\beta\gamma$-systems are strongly finitely generated.

In [20] we initiated a similar study of the invariant subalgebras of the rank n Heisenberg vertex algebra $\mathcal{H}(n)$. The full automorphism group of $\mathcal{H}(n)$ preserving a natural conformal structure is the orthogonal group $O(n)$. Motivated by classical invariant theory, we conjectured that $\mathcal{H}(n)^{O(n)}$ is a $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, \ldots, n^2+3n)$. For $n = 1$, this was already known to Dong–Nagatomo [5], and we proved it for $n = 2$ and $n = 3$. We also showed that this conjecture implies the strong finite generation of $\mathcal{H}(n)^G$ for an arbitrary reductive group G; see Theorem 6.9 of [20].

In this paper, we study invariant subalgebras of the universal affine vertex algebra $V_k(\mathfrak{g}, B)$ for a finite-dimensional Lie algebra $\mathfrak{g}$ equipped with a nondegenerate, symmetric, invariant bilinear form B. In the special case where $\mathfrak{g}$ is simple and B is the normalized Killing form, it is customary to denote $V_k(\mathfrak{g}, B)$ by $V_k(\mathfrak{g})$. Recall that $V_k(\mathfrak{g}, B)$ has generators X^ξ, which are linear in $\xi \in \mathfrak{g}$, and satisfy the OPE relations

$$X^\xi(z)X^\eta(w) \sim kB(\xi, \eta)(z - w)^{-2} + X^{[\xi,\eta]}(w)(z - w)^{-1}.$$

Let G be a reductive group of automorphisms of $V_k(\mathfrak{g}, B)$ for all $k \in \mathbb{C}$. Our main result is the following.

Theorem 1.1. *For any $\mathfrak{g}$, B, and G, $V_k(\mathfrak{g}, B)^G$ is strongly finitely generated for generic values of k, i.e., for $k \in \mathbb{C} \setminus K$ where K is at most countable.*

Note that when $\mathfrak{g}$ is abelian and $k \neq 0$, $V_k(\mathfrak{g}, B)^G \cong \mathcal{H}(n)^G$ for $n = \dim(\mathfrak{g})$, so this result both improves and generalizes our earlier study of the vertex algebras $\mathcal{H}(n)^G$. The proof of Theorem 1.1 is divided into three steps. The first step is to prove it in the special case where $\mathfrak{g}$ is abelian and $G = O(n)$. We will show that $\mathcal{H}(n)^{O(n)}$ is of type $\mathcal{W}(2, 4, \ldots, n^2 + 3n)$ for $n \leq 6$, and although we do not prove this conjecture in general, we will establish the strong finite generation of $\mathcal{H}(n)^{O(n)}$ for all n. The second step (which is a minor modification of