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Koszul property of projections of the Veronese cubic surface

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Dedicated to Tito Valla, our teacher and friend

Abstract

Let $V \subset \mathbf{P}^9$ be the Veronese cubic surface. We classify the projections of V to $\mathbf{P}^8$ whose coordinate rings are Koszul. In particular we obtain a purely theoretical proof of the Koszulness of the pinched Veronese, a result obtained originally by Caviglia using filtrations, deformations and computer assisted computations. To this purpose we extend, to certain complete intersections, results of Conca, Herzog, Trung and Valla concerning homological properties of diagonal algebras.

MSC: 13D02; 14M99

Keywords: Koszul algebras; Projections of the Veronese surface; Diagonal algebras; Complete intersections

1. Introduction

Koszul algebras were originally introduced by Priddy [18] in his study of homological properties of graded (non-commutative) algebras arising from various constructions in algebraic topology. Given a field K, a positively graded K-algebra $A = \oplus_{i \in \mathbf{N}} A_i$ with $A_0 = K$ is Koszul if the field K, viewed as a A-module via the identification $K = A/A_+$, has a linear free resolution.

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In the very interesting volume [17] Polishchuk and Positselski discuss various surprising aspects of Koszulness.

In the commutative setting Koszul algebras can be characterized by means of the relative Castelnuovo–Mumford regularity. Paraphrasing Hochster [13, p. 887], one can say that life is worth living in standard graded algebras when all the finitely generated graded modules have finite (relative) Castelnuovo–Mumford regularity. Such algebras are exactly the commutative Koszul algebras; see Avramov and Eisenbud [3] and Avramov and Peeva [4].

Let K be a field and $R = K[x_0, x_1, x_2]$. The pinched Veronese is the K-subalgebra of R generated by all the monomials of degree 3 with the exception of $x_0x_1x_2$. It can be seen as the coordinate ring of a projection from a point of the Veronese cubic surface $V_{2,3}$ of $\mathbf{P}^9$, that is, the embedding of $\mathbf{P}^2$ in $\mathbf{P}^9$ with the forms of degree 3. It is a "generic" projection with respect to the stratification of the ambient space by the secant varieties of $V_{2,3}$. That is, the center of the projection is outside the second secant variety $\sec_2(V_{2,3})$ of $V_{2,3}$ and the third secant is the ambient space $\mathbf{P}^9$.

In the nineties Sturmfels asked, in a conversation with Peeva, whether the pinched Veronese is Koszul and the problem became quickly known as a benchmark example to test new theorems and techniques. The first author of the present paper proved in [5] that the pinched Veronese is indeed Koszul by using a combination of arguments based on filtrations, deformations and computer assisted computations.

More generally, one can ask the same question for any projection of $V_{2,3}$ to $\mathbf{P}^8$. In particular, one can ask whether the projection of $V_{2,3}$ to $\mathbf{P}^8$ from a point that does not belong to $\sec_2(V_{2,3})$ is Koszul. The goal of the paper is to show that this is indeed the case. As a special case, we obtain a entirely theoretical proof of the Koszulness of the pinched Veronese.

To achieve this result we develop in Section 2 homological arguments that generalize results of Conca, Herzog, Trung and Valla [7]. Given a standard $\mathbf{Z}^2$-graded K-algebra S and a cyclic subgroup Δ of $\mathbf{Z}^2$ one considers the "diagonal" subalgebra S_Δ of S defined as $\oplus_{v\in\Delta} S_v$. Similarly, for every $\mathbf{Z}^2$-graded S-module M one defines the S_Δ-module M_Δ as $\oplus_{v\in\Delta} M_v$. For every element $w \in \mathbf{Z}^2$, the shifted copy $S(w)$ of S is defined as the $\mathbf{Z}^2$-graded S module whose v-th component is S_{v+w}.

In the transfer of homological information from S to S_Δ it is crucial to bound the homological invariants of the shifted-diagonal modules $S(w)_\Delta$ as S_Δ-modules. When S is a bigraded polynomial ring, it is proved in [7] that the modules $S(w)_\Delta$ have a linear S_Δ-free resolution. We extend this result to the main diagonal $\Delta = (1, 1)\mathbf{Z}$ of certain bigraded complete intersections; see Section 2.

In Section 3 we prove that, given a complete intersection I of 3 quadrics in a polynomial ring R, the K-subalgebra of R generated by the cubics in I is Koszul. This is done by constructing a complex whose homology vanishes along the relevant diagonal. Finally in Section 4 we reinterpret the result of Section 3 to get the classification of the projections to $\mathbf{P}^8$ of the cubic Veronese surface with a Koszul coordinate ring.

2. Generalities and preliminary results

Let K be a field and A be a standard graded K-algebra, that is, a commutative algebra of the form $A = \oplus_{i\in\mathbf{N}} A_i$ such that $A_0 = K$, $\dim_K A_1$ is finite and A is generated as a K-algebra by A_1. In other words, a standard graded K-algebra can be written as $A = S/I$ where $S = K[x_1, \dots, x_n]$ is a polynomial ring over K equipped with the graded structure induced by the assignment $\deg(x_i) = 1$ for every i and I is a homogeneous ideal. The algebra A is said to