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A criterion for the integrality of the Taylor coefficients of mirror maps in several variables

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Abstract

We give a necessary and sufficient condition for the integrality of the Taylor coefficients at the origin of formal power series $q_i(\mathbf{z}) = z_i \exp(G_i(\mathbf{z})/F(\mathbf{z}))$, with $\mathbf{z} = (z_1, \dots, z_d)$ and where $F(\mathbf{z})$ and $G_i(\mathbf{z}) + \log(z_i)F(\mathbf{z})$, $i = 1, \dots, d$ are particular solutions of certain A-systems of differential equations. This criterion is based on the analytical properties of Landau's function (which is classically associated with sequences of factorial ratios) and it generalizes the criterion in the case of one variable presented in [E. Delaygue, Critère pour l'intégralité des coefficients de Taylor des applications miroir, J. Reine Angew. Math. 662 (2012) 205–252]. One of the techniques used to prove this criterion is a generalization of a version of a theorem of Dwork on formal congruences between formal series, proved by Krattenthaler and Rivoal in [C. Krattenthaler, T. Rivoal, Multivariate *p-adic* formal congruences and integrality of Taylor coefficients of mirror maps, in: L. Di Vizio, T. Rivoal (Eds.), Théories Galoisiennes et Arithmétiques des Équations Différentielles, in: Séminaire et Congrés, vol. 27, Soc. Math. France, Paris, 2011, pp. 279–307].

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1. Introduction

The mirror maps considered in this article are formal series of d variables $z_i(\mathbf{x})$, $i = 1, \dots, d$, with $\mathbf{x} = (x_1, \dots, x_d)$. The map $\mathbf{x} \mapsto (z_1(\mathbf{x}), \dots, z_d(\mathbf{x}))$ is the compositional inverse of the

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map $\mathbf{x} \mapsto (q_1(\mathbf{x}), \dots, q_d(\mathbf{x}))$, with $q_i(\mathbf{x}) = x_i \exp(G_i(\mathbf{x})/F(\mathbf{x}))$ and where $F(\mathbf{x})$ and $G_i(\mathbf{x}) + \log(x_i)F(\mathbf{x})$ are particular solutions of a certain A-system of linear differential equations. These objects are geometric in nature because the series $F(\mathbf{x})$ are A-hypergeometric functions ([1]) which can be viewed as the period of certain multi-parameter families of algebraic varieties in a product of weighted projective spaces (see [6] for details).

A classical example of multivariate mirror maps, studied in [2,13,9] is related to the series

$$F(z_1, z_2) = \sum_{m,n\geq 0} \frac{(3m+3n)!}{m!^3 n!^3} z_1^m z_2^n, \tag{1.1}$$

which is a solution of the system of differential equations

$$\begin{cases} D_1^3 y - z_1 (3D_1 + 3D_2 + 1)(3D_1 + 3D_2 + 2)(3D_1 + 3D_2 + 3)\, y = 0, \\ D_2^3 y - z_2 (3D_1 + 3D_2 + 1)(3D_1 + 3D_2 + 2)(3D_1 + 3D_2 + 3)\, y = 0, \end{cases}$$

where $D_1 = z_1 \frac{d}{dz_1}$ and $D_2 = z_2 \frac{d}{dz_2}$. We find two other solutions of this system $G_1(z_1, z_2) + \log(z_1)F(z_1, z_2)$ and $G_2(z_1, z_2) + \log(z_2)F(z_1, z_2)$ where

$$G_1(z_1, z_2) = \sum_{m,n\geq 0} \frac{(3m+3n)!}{m!^3 n!^3} (3H_{3m+3n} - 3H_m) z_1^m z_2^n$$

and $G_2(z_1, z_2) = G_1(z_2, z_1)$. This set of solutions enables us to define two *canonical coordinates* $q_1(z_1, z_2) = z_1 \exp(G_1(z_1, z_2)/F(z_1, z_2))$ and $q_2(z_1, z_2) = z_2 \exp(G(z_1, z_2)/F(z_1, z_2))$.

The associated *mirror maps* are defined by the formal series $z_1(q_1, q_2)$ and $z_2(q_1, q_2)$ such that the map $(q_1, q_2) \mapsto (z_1(q_1, q_2), z_2(q_1, q_2))$ is the compositional inverse of the map $(z_1, z_2) \mapsto (q_1(z_1, z_2), q_2(z_1, z_2))$.

According to Corollary 1 of [9], the series $q_1(z_1, z_2), q_2(z_1, z_2), z_1(q_1, q_2)$ and $z_2(q_1, q_2)$ have integral Taylor coefficients.

Mirror maps are of interest in Mathematical Physics and Algebraic Geometry. In particular, within Mirror Symmetry Theory, it has been observed that the Taylor coefficients of mirror maps are integers. This surprising observation has led to the study of these objects within Number Theory, which has led to its proof in many cases (see further down in the introduction). The aim of this article is to establish a necessary and sufficient condition for the integrality of all the Taylor coefficients of mirror maps defined by ratios of factorials of linear forms.

1.1. *Definition of mirror maps*

In order to define the mirror maps considered in this article, we introduce some standard multi-index notation, which we use throughout the article. Namely, given a positive integer d, $k \in \{1, \dots, d\}$ and vectors $\mathbf{m} := (m_1, \dots, m_d)$ and $\mathbf{n} := (n_1, \dots, n_d)$ in $\mathbb{R}^d$, we write $\mathbf{m} \cdot \mathbf{n}$ for the scalar product $m_1 n_1 + \cdots + m_d n_d$ and $\mathbf{m}^{(k)}$ for m_k. We write $\mathbf{m} \geq \mathbf{n}$ if and only if $m_i \geq n_i$ for all $i \in \{1, \dots, d\}$. In addition, if $\mathbf{z} := (z_1, \dots, z_d)$ is a vector of variables and if $\mathbf{n} := (n_1, \dots, n_d) \in \mathbb{Z}^d$, then we write $\mathbf{z}^{\mathbf{n}}$ for the product $z_1^{n_1} \cdots z_d^{n_d}$. Finally, we write $\mathbf{0}$ for the vector $(0, \dots, 0) \in \mathbb{Z}^d$.

Given two sequences of vectors in $\mathbb{N}^d$, $e := (\mathbf{e}_1, \dots, \mathbf{e}_{q_1})$ and $f := (\mathbf{f}_1, \dots, \mathbf{f}_{q_2})$, we write $|e| := \sum_{i=1}^{q_1} \mathbf{e}_i$ and $|f| := \sum_{i=1}^{q_2} \mathbf{f}_i \in \mathbb{N}^d$ so that, for all $k \in \{1, \dots, d\}$, we have $|e|^{(k)} = \sum_{i=1}^{q_1} \mathbf{e}_i^{(k)}$

[1] The A-hypergeometric series are also called GKZ hypergeometric series. See [13] for an introduction to these series, which generalize the classical hypergeometric series to the multivariate case.