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# Extensions of smooth mappings into biduals and weak continuity

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**Abstract**

We study properties of uniformly differentiable mappings between real Banach spaces. Among our main results are generalizations of a number of classical results for linear operators on $\mathcal{L}_\infty$-spaces into the setting of uniformly differentiable mappings. Denote by $B_X$ the closed unit ball of a Banach space $X$. Let $X$ be a $\mathcal{L}_{\infty,\lambda}$-space, $\lambda \geq 1$, and let $Y$ be a Banach space. Let $T : B_X \to Y$ be a continuous mapping which is uniformly differentiable in the open unit ball of $X$. Assuming that $T$ is weakly compact, then $T$ can be extended, preserving its best smoothness properties, into the mapping from the $\frac{1}{\lambda}$-multiple of the unit ball of any superspace of the domain space $X$ into the same range space $Y$. We also show that $T$ maps weakly Cauchy sequences from $\lambda B_X$ into norm convergent sequences in $Y$. This is a uniformly smooth version of the Dunford–Pettis property for the $\mathcal{L}_{\infty,\lambda}$-spaces. We also show that a uniformly differentiable mapping $T$, which is not necessarily weakly compact, still maps weakly Cauchy sequences from $\lambda B_X$ into norm convergent sequences in $Y$, provided $Y^{**}$ does not contain an isomorphic copy of $c_0$.

We prove that for certain pairs of Banach spaces the completion of the space of polynomials equipped with the topology of uniform convergence on the bounded sets (of the functions and their derivatives up to order $k$) coincides with the space of uniformly differentiable (up to order $k$) mappings.

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Our work is based on a number of tools that are of independent interest. We prove, for every pair of Banach spaces $X, Y$, that any continuous mapping $T : B_X \to Y$, which is uniformly differentiable of order up to $k$ in the interior of $B_X$, can be extended, preserving its best smoothness, into a bidual mapping $\tilde{T} : B_{X^{**}} \to Y^{**}$. Another main tool is a result of Zippin's type. We show that weakly Cauchy sequences in $X = C(K)$ can be uniformly well approximated by weakly Cauchy sequences from a certain $C[0, \alpha]$, $\alpha$ is a countable ordinal, subspace of $X^{**}$.


*MSC:* 46B03; 46B10

*Keywords:* Extension to biduals; Dunford–Pettis property; Smoothness; Approximation by polynomials; Reduction lemma

## 1. Introduction

In the present note we study the properties of uniformly differentiable mappings between Banach spaces. Of particular importance for us are their weak sequential continuity properties. We give generalizations of a number of classical results for linear operators on $\mathcal{L}_\infty$-spaces into the setting of uniformly differentiable mappings. Let us give a few examples. Let $X$ be a $\mathcal{L}_{\infty,\lambda}$-space, $\lambda \geq 1$, and let $Y$ be a Banach space. Let $T : B_X \to Y$ be a weakly compact, uniformly smooth (in the interior of $B_X$) operator. Then $T$ can be extended, preserving its best smoothness properties, into the mapping from the $\frac{1}{\lambda}$-multiple of the unit ball of any superspace. This is a generalization of the work of Lindenstrauss in [53]. We show that $T$ maps weakly Cauchy sequences into norm convergent ones. This is a smooth version of the Dunford–Pettis property for the $\mathcal{L}_\infty$-spaces, generalizing the work of Dunford, Pettis, Grothendieck, Lindenstrauss and Pełczyński, and Ryan who proved the result for polynomials.

We also show that a uniformly differentiable mapping $T$, which is not necessarily weakly compact, still maps weakly Cauchy sequences from $\lambda B_X$ into norm convergent sequences in $Y$, provided $Y^{**}$ does not contain an isomorphic copy of $c_0$. If we add the assumption that $X$ contains no copy of $\ell_1$ then $T$ is always weakly compact provided $Y^{**}$ does not contain an isomorphic copy of $c_0$, generalizing Pełczyński's results for polynomial mappings. Other applications involve a generalization of the Stone–Weierstrass theorem. We prove that for certain pairs of Banach spaces the completion of the space of polynomials equipped with the topology of uniform convergence on the bounded sets (of the functions and their derivatives up to order $k$) coincides with the space of uniformly differentiable (up to order $k$) mappings. This is a generalization of the classical de la Vallée Poussin theorem, and the work of Aron and Prolla.

Our work is based on a number of tools, which are of independent interest. We prove that any uniformly differentiable mapping (of order up to $k$) $T : B_X \to Y$ can be extended, preserving its best smoothness, into a bidual mapping $\tilde{T} : B_{X^{**}} \to Y^{**}$. This is a generalization of the classical work of Arens [6], Aron and Berner [10] in the case of polynomials (or holomorphic functions). We give a detailed proof of this extension, including one of its main ingredients—the Converse Taylor theorem, which is again of independent interest (we thank M. Johanis for allowing us to reproduce his proof of this result) and which has not appeared in the literature in this generality. Another main tool is a result of Zippin's type. We show that any weakly Cauchy sequence in $X = C(K)$ can be uniformly well approximated (if $X$ is considered as a canonical subspace of $X^{**}$) by a weakly Cauchy sequence from a certain subspace of $X^{**}$, which is isometric to $C[0, \alpha]$, $\alpha$ countable ordinal. Furthermore, we show that if a Banach space $X$ does not contain a copy of $\ell_1$ then the weak (resp. uniformly weak) continuity of $T : B_X \to Y$ is a sequential