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Random polarizations

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Abstract

We derive conditions under which random sequences of polarizations (two-point symmetrizations) on $\mathbb{S}^d$, $\mathbb{R}^d$, or $\mathbb{H}^d$ converge almost surely to the symmetric decreasing rearrangement. The parameters for the polarizations are independent random variables whose distributions need not be uniform. The proof of convergence hinges on an estimate for the expected distance from the limit that yields a bound on the rate of convergence. In the special case of i.i.d. sequences, almost sure convergence holds even for polarizations chosen at random from suitable small sets. As corollaries, we find bounds on the rate of convergence of Steiner symmetrizations that require no convexity assumptions, and show that full rotational symmetry can be achieved by randomly alternating Steiner symmetrizations in a finite number of directions that satisfy an explicit non-degeneracy condition. We also present some negative results on the rate of convergence and give examples where convergence fails.

Keywords: Geometric inequalities; Two-point symmetrization; Steiner symmetrization; Symmetric decreasing rearrangement; Almost sure convergence; Rate of convergence; Power-law bound; Hausdorff metric

1. Introduction

Many classical geometric inequalities were proved by first establishing the inequality for a simple geometric transformation, such as Steiner symmetrization or polarization. Steiner symmetrization is a volume-preserving rearrangement that introduces a reflection symmetry, and polarization pushes mass across a hyperplane towards the origin. (Proper definitions will

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be given below.) To mention just a few examples, there are proofs of the isoperimetric inequality and Santaló's inequality based on the facts that Steiner symmetrization reduces perimeter [26,10] and increases the Mahler product [23]. Inequalities for capacities and path integrals follow from the observation that polarization increases convolution functionals [30,13,1,2] and related multiple integrals [9,24,25]. This approach reduces the geometric inequalities to one-dimensional problems (in the case of Steiner symmetrization) or even to combinatorial identities (in the case of polarization). It can also be exploited to characterize equality cases [4,3,9]. A major point is to construct sequences of the simple rearrangements that produce full rotational symmetry in the limit.

In this paper, we study the convergence of random sequences of polarizations to the symmetric decreasing rearrangement. The result of n random polarizations of a function f is denoted by $S_{W_1 \dots W_n} f$, where each W_i is a random variable that determines a reflection. We assume that the W_i are independent, but not necessarily identically distributed, and derive conditions under which

$$S_{W_1 \dots W_n} f \longrightarrow f^* \quad (n \to \infty) \quad \text{almost surely.} \tag{1.1}$$

Rearrangements have been studied in many different spaces, with various notions of convergence. We work with continuous functions in the topology of uniform convergence, while most classical results are stated for compact sets with the Hausdorff metric. These notions of convergence turn out to be largely equivalent because of the monotonicity properties of rearrangements.

For sequences of Steiner symmetrizations along uniformly distributed random directions, convergence is well known [22,29]. It has recently been shown that certain uniform geometric bounds on the distributions guarantee convergence for a broad class of rearrangements that includes polarization, Steiner symmetrization, the Schwarz rounding process, and the spherical cap symmetrization [27]. Among these rearrangements, polarization plays a special role, because it is elementary to define, easy to use, and can approximate the others. Our conditions for convergence allow the distribution of the W_i to be far from uniform. We also prove bounds on the rate of convergence, and show how convergence can fail. Our results shed new light on Steiner symmetrizations. In particular, we obtain bounds on the rate of convergence for Steiner symmetrizations of arbitrary compact sets.

2. Main results

Let $\mathbb{X}$ be either the sphere $\mathbb{S}^d$, Euclidean space $\mathbb{R}^d$, or the standard hyperbolic space $\mathbb{H}^d$, equipped with the uniform Riemannian distance $d(x, y)$, the Riemannian volume $m(A)$, and a distinguished point $o \in \mathbb{X}$, which we call the origin. The ball of radius ρ about a point $x \in \mathbb{X}$ is denoted by $B_\rho(x)$; if the center is at $x = o$ we simply write B_ρ. We denote by $\operatorname{dist}(x, A) = \inf_{y \in A} d(x, y)$ the distance between a point and a set, and by

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \operatorname{dist}(x, B), \sup_{x \in B} \operatorname{dist}(x, A) \right\}$$

the **Hausdorff distance** between two sets.

If A is a set of finite volume in $\mathbb{X}$, we denote by A^* the open ball centered at the origin with $m(A^*) = m(A)$. We consider nonnegative measurable functions f on $\mathbb{X}$ that vanish weakly at infinity, in the sense that the level sets $\{x : f(x) > t\}$ have finite volume for all $t > 0$. (On the sphere, this condition is empty.) The **symmetric decreasing rearrangement** f^* is the unique