Available online at www.sciencedirect.com
SciVerse ScienceDirect
Advances in Mathematics 234 (2013) 574–617
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Finite order spreading models

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Received 28 January 2011; accepted 6 November 2012
Available online 8 December 2012

Communicated by the Managing Editors of AIM

Abstract

Extending the classical notion of spreading model, the k-spreading models of a Banach space are introduced, for every $k \in \mathbb{N}$. The definition, which is based on the k-sequences and plegma families, reveals a new class of spreading sequences associated to a Banach space. Most of the results of the classical theory are stated and proved in the higher order setting. Moreover, new phenomena like the universality of the class of the 2-spreading models of c_0 and the composition property are established. As consequence, a problem concerning the structure of the k-iterated spreading models is solved.

MSC: 46B03; 46B06; 46B25; 46B45; 05D10

Keywords: Spreading models; Ramsey theory

0. Introduction

The present work was motivated by a problem of E. Odell and Th. Schlumprecht concerning the structure of the k-iterated spreading models of the Banach spaces. Our attempt to answer the problem led to the k-spreading models which in turn are based on the k-sequences and plegma families. The aim of this paper is to introduce the above concepts and to develop a theory yielding, among others, a solution to the aforementioned problem.

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.11.004

Spreading models, invented by A. Brunel and L. Sucheston (c.f. [8]), possess a key role in the modern Banach space theory. Let us recall that a spreading model of a Banach space X is a spreading sequence[1] generated (in a sense that will be explained later on) by a sequence of X. The spreading sequences have regular structure and the spreading models act as the tool for realizing that structure in the space X in an asymptotic manner. This together with the Brunel–Sucheston's discovery that every bounded sequence has a subsequence generating a spreading model determine the significance and importance of this concept. For a comprehensive presentation of the theory of the spreading models we refer the interested reader to the monograph of B. Beauzamy and J.-T. Lapresté (c.f. [6]).

Iteration is naturally applicable to spreading models. Thus one could define the 2-iterated spreading models of a Banach space X to be the spreading sequences which occur as spreading models of the spaces generated by spreading models of X. Further iteration yields the k-iterated spreading models of X, for every $k \in \mathbb{N}$. Iterated spreading models appeared in the literature shortly after Brunel–Sucheston's invention. Indeed, B. Beauzamy and B. Maurey in [7], answering a problem of H.P. Rosenthal, showed that the class of the 2-iterated spreading models do not coincide with the corresponding one of the spreading models. In particular they constructed a Banach space admitting the usual basis of ℓ_1 as a 2-iterated spreading model and not as a spreading model.

E. Odell and Th. Schlumprecht in [19] asked whether or not every Banach space admits a k-iterated spreading model equivalent to the usual basis of ℓ_p, for some $1 \leq p < \infty$, or c_0. Let us also point out that in the same paper they provided a reflexive space $\mathfrak{X}$ with an unconditional basis such that no ℓ_p or c_0 is embedded into the space generated by any spreading model of the space. This remarkable result answered a long standing problem of the Banach space theory and is in contrast with Krivine's theorem [15], although both are related to the structure of finite dimensional subspaces of X.

Our approach uses the notion of k-spreading models which in many cases include the k-iterated ones. The k-spreading models are always spreading sequences $(e_n)_n$ in a seminormed space E. They are generated by k-sequences $(x_s)_{s\in[\mathbb{N}]^k}$, where $[\mathbb{N}]^k$ denotes the family of all k-subsets of $\mathbb{N}$. A critical ingredient in the definition is the plegma[2] families $(s_i)_{i=1}^l$ of elements of $[\mathbb{N}]^k$, described as follows.

A finite sequence $(s_j)_{j=1}^l$ in $[\mathbb{N}]^k$ is a plegma family if its elements satisfy the following order relation: for every $1 \leq i \leq k$, $s_1(i) < \cdots < s_l(i)$ and for every $1 \leq i < k$, $s_l(i) < s_1(i+1)$. The plegma families, as they are used in the definition, force a weaker asymptotic relation of the k-spreading models to the space X, as k increases. For $k = 1$, the plegma families coincide to the finite subsets of $\mathbb{N}$ yielding that the new definition of the 1-spreading models recovers the classical one. For $k > 1$, the plegma families have a quite strict behavior which is described in Section 1 of the paper. Of independent interest is also Lemma 2 stated below.

The k-spreading models of a Banach space X are denoted by $\mathcal{SM}_k(X)$ and these sets define an increasing sequence. As the definition easily yields, the same holds for the k-iterated ones.

[1] A sequence $(e_n)_n$ in a seminormed space $(E, \|\cdot\|_*)$ is called spreading if for every $n \in \mathbb{N}$, $k_1 < \cdots < k_n$ in $\mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$ we have that $\|\sum_{j=1}^n a_j e_j\|_* = \|\sum_{j=1}^n a_j e_{k_j}\|_*$. In the literature the term "spreading model" usually indicates the space generated by the corresponding spreading sequence rather than the sequence itself. We have chosen to use the term for the spreading sequence and whenever we refer to ℓ_p or c_0 spreading model we shall mean that the spreading sequence is equivalent to the usual basis of the corresponding space.

[2] The name comes from the Greek word "$\pi\lambda\varepsilon\gamma\mu\alpha$" which in English corresponds to the word "grid".