Available online at www.sciencedirect.com
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Advances in Mathematics 234 (2013) 667–696
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Concavification of free entropy

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Received 5 March 2012; accepted 6 November 2012
Available online 13 December 2012

Communicated by Dan Voiculescu

Abstract

We introduce a modification of Voiculescu's free entropy which coincides with the lim inf variant of Voiculescu's free entropy on extremal states, but is a concave upper semi-continuous function on the trace state space. We also extend the orbital free entropy of Hiai et al. (2009) [8] to non-hyperfinite multivariables and prove freeness in the case of additivity of Voiculescu's entropy (or vanishing of our extended orbital entropy).

MSC: primary 46L54; secondary 60G15; 94A17

Keywords: Free entropy; Orbital entropy; Additivity of entropy; Poulsen simplex; Concentration of measure; Csiszar's projection

1. Introduction

Voiculescu has introduced a free entropy quantity, for tracial states on a von Neumann algebra generated by n self-adjoint elements, which has been very useful for the solution of many long standing open problems in von Neumann algebra theory. It turns out that free entropy satisfies an unusual property for an entropy quantity which is a "degenerate convexity" property, i.e. the entropy of any nonextremal state is $-\infty$, which is in sharp contrast with the usual concavity and

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.11.003

upper semi-continuity property of classical entropy. Recently Hiai [7] defined a free analogue of pressure and considered its Legendre transform. He obtained a quantity which is concave and upper semi-continuous, and majorizes Voiculescu's free entropy. It is not clear whether this quantity coincides with Voiculescu's free entropy on extremal states. In this paper we introduce a modified definition, through random matrix approximations, which yields a quantity which is both concave upper semi-continuous, and coincides with the lim inf variant of Voiculescu's free entropy on extremal states. Our main argument is the simple observation that a probability measure on a compact convex set, whose barycenter is close to an extremal point, has most of its mass concentrated near this point (see Lemma 6.1 below). This is obvious in finite dimension, but requires further clarification in infinite dimension. In this paper we rely on the fact that the convex set we consider is a Poulsen simplex.

We use an analogous idea to generalize the definition of free orbital entropy, due to Hiai, Miyamoto and Ueda [8]. In this paper, the authors introduced, via a microstates approach, an entropy quantity $\chi_{orb}(\mathbf{X_1}, \ldots, \mathbf{X_n})$, where each $\mathbf{X_i}$ is a finite set of noncommutative random variables generating a hyperfinite algebra. They used this quantity to generalize Voiculescu's additivity result [20], namely: for noncommutative random variables $X_1, \ldots, X_n$, if

$$\chi(X_1, \ldots, X_n) = \chi(X_1) + \cdots + \chi(X_n)$$

and these quantities are finite, then the X_i are free. More generally, they showed that $\chi_{orb}(\mathbf{X_1}, \ldots, \mathbf{X_n}) = 0$ is equivalent to freeness in the hyperfinite context above even though the finiteness of entropy fails in general in this case. They recover the previous result since they also show:

$$\chi(X_1, \ldots, X_n) = \chi_{orb}(X_1, \ldots, X_n) + \chi(X_1) + \cdots + \chi(X_n),$$

in case these quantities are finite.

In Section 7, we introduce a definition of $\chi_{orb}(\mathbf{X_1}, \ldots, \mathbf{X_n})$, for arbitrary finite sets $\mathbf{X_i}$ of noncommutative random variables, obtained by replacing microstates by probability measures. We show that many of the arguments of [8] have analogues in this setting, and we obtain the full generalization of the additivity result when random variables X_i are replaced by arbitrary finite sets $\mathbf{X_i}$.

This paper is organized as follows. We start by recalling some well known facts on trace states and on Legendre transform and classical entropy (including Csiszar's projections result) in Sections 2 and 3. Then we prove the main result about concavification in Sections 4 and 6, after a few preliminaries about Poulsen simplices in Section 5. In Section 7 we extend the definition of orbital entropy, and prove freeness in the case of additivity of Voiculescu's entropy, in Corollary 7.4. Finally, after a few more preliminaries in Section 8, Section 9 is devoted to some further variants and extensions of our definitions, which might prove useful for future applications.

2. The set of trace states

Let $\mathbf{C}\langle X_1, \ldots, X_n\rangle$ be the free $*$-algebra with unit generated by $n \geq 1$ self-adjoint elements $X_1, \ldots, X_n$, which we identify with the space of noncommutative polynomials in the indeterminates $X_1, \ldots, X_n$. We consider the set $\mathcal{S}_c^n$ of trace states on $\mathbf{C}\langle X_1, \ldots, X_n\rangle$. This set consists in all positive, tracial $*$-linear maps $\tau : \mathbf{C}\langle X_1, \ldots, X_n\rangle \to \mathbf{C}$ such that $\tau(1) = 1$ and, for any $P \in \mathbf{C}\langle X_1, \ldots, X_n\rangle$ there exists some constant $R_P > 0$ such that

$$\tau((P^*P)^k) \leq R_P^{2k} \quad \text{for } k \geq 0 \tag{2.1}$$