Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 234 (2013) 697–734

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Transversality family of expanding rational semigroups

Hiroki Sumi[a,*], Mariusz Urbański[b]

[a] *Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama, Toyonaka, Osaka, 560-0043, Japan*

[b] *Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA*

Received 10 February 2012; accepted 30 October 2012

Communicated by Kenneth Falconer

Abstract

We study finitely generated expanding semigroups of rational maps with overlaps on the Riemann sphere. We show that if a d-parameter family of such semigroups satisfies the transversality condition, then for almost every parameter value the Hausdorff dimension of the Julia set is the minimum of 2 and the zero of the pressure function. Moreover, the Hausdorff dimension of the exceptional set of parameters is estimated. We also show that if the zero of the pressure function is greater than 2, then typically the 2-dimensional Lebesgue measure of the Julia set is positive. Some sufficient conditions for a family to satisfy the transversality conditions are given. We give non-trivial examples of families of semigroups of non-linear polynomials with the transversality condition for which the Hausdorff dimension of the Julia set is typically equal to the zero of the pressure function and is less than 2. We also show that a family of small perturbations of the Sierpinski gasket system satisfies that for a typical parameter value, the Hausdorff dimension of the Julia set (limit set) is equal to the zero of the pressure function, which is equal to the similarity dimension. Combining the arguments on the transversality condition, thermodynamical formalisms and potential theory, we show that for each $a \in \mathbb{C}$ with $|a| \neq 0, 1$, the family of small perturbations of the semigroup generated by $\{z^2, az^2\}$ satisfies that for a typical parameter value, the 2-dimensional Lebesgue measure of the Julia set is positive.

MSC: primary 37F35; secondary 37F15

* Corresponding author.
E-mail addresses: sumi@math.sci.osaka-u.ac.jp (H. Sumi), urbanski@unt.edu (M. Urbański).
URLs: http://www.math.sci.osaka-u.ac.jp/~sumi/ (H. Sumi), http://www.math.unt.edu/~urbanski/ (M. Urbański).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.10.020

Keywords: Complex dynamical systems; Rational semigroups; Expanding semigroups; Julia set; Transversality condition; Hausdorff dimension; Bowen parameter; Random complex dynamics; Random iteration; Iterated function systems with overlaps; Self-similar sets

1. Introduction

A *rational semigroup* is a semigroup generated by a family of non-constant rational maps $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, with the semigroup operation being functional composition. A polynomial semigroup is a semigroup generated by a family of non-constant polynomial maps on $\hat{\mathbb{C}}$. The work on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin [8], who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups of Möbius transformations, and by F. Ren's group [44], who studied such semigroups from the perspective of random dynamical systems.

The theory of the dynamics of rational semigroups on $\hat{\mathbb{C}}$ has developed in many directions since the 1990s [8,44,22,24–30,39,31,32,23,33–37]. We recommend [22] as an introductory article. For a rational semigroup G, we denote by $F(G)$ the maximal open subset of $\hat{\mathbb{C}}$ where G is normal. The set $F(G)$ is called the Fatou set of G. The complement $J(G) := \hat{\mathbb{C}} \setminus F(G)$ is called the Julia set of G. Since the Julia set $J(G)$ of a rational semigroup $G = \langle f_1, \dots, f_m \rangle$ generated by finitely many elements $f_1, \dots, f_m$ has *backward self-similarity* i.e.

$$J(G) = f_1^{-1}(J(G)) \cup \cdots \cup f_m^{-1}(J(G)), \tag{1.1}$$

(see [24,26]), rational semigroups can be viewed as a significant generalization and extension of both the theory of iteration of rational maps (see [14,2]) and conformal iterated function systems (see [11]). Indeed, because of (1.1), the analysis of the Julia sets of rational semigroups somewhat resembles "backward iterated functions systems", however since each map f_j is not in general injective (critical points), some qualitatively different extra effort in the case of semigroups is needed. The theory of the dynamics of rational semigroups borrows and develops tools from both of these theories. It has also developed its own unique methods, notably the skew product approach (see [26–29,31,38,32,34–37,40,39,41]).

The theory of the dynamics of rational semigroups is intimately related to that of the random dynamics of rational maps. The first study of random complex dynamics was given in [6]. In [3,7], random dynamics of quadratic polynomials were investigated. The paper [12] develops the thermodynamic formalism of random distance expanding maps and, in particular, applies it to random polynomials. The deep relation between these fields (rational semigroups, random complex dynamics, and (backward) IFS) is explained in detail in the subsequent papers [30,31,38,32–37] of the first author. For a random dynamical system generated by a family of polynomial maps on $\hat{\mathbb{C}}$, let $T_\infty : \hat{\mathbb{C}} \to [0, 1]$ be the function of probability of tending to $\infty \in \hat{\mathbb{C}}$. In [34,36,37] it was shown that under certain conditions, T_∞ is continuous on $\hat{\mathbb{C}}$ and varies only on the Julia set of the associated rational semigroup (further results were announced in [35]). For example, for a random dynamical system in Remark 1.5, T_∞ is continuous on $\hat{\mathbb{C}}$ and the set of varying points of T_∞ is equal to the Julia set of Fig. 1, which is a thin fractal set with Hausdorff dimension strictly less than 2. From this point of view also, it is very interesting and important to investigate the figure and the dimension of the Julia sets of rational semigroups.

In this paper, for an expanding finitely generated rational semigroup $\langle f_1, \dots, f_m \rangle$, we deal at length with the relation between the Bowen parameter $\delta(f)$ (the unique zero of the pressure