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Idempotent transformations of finite groups

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Abstract

We describe the action of idempotent transformations on finite groups. We show that finiteness is preserved by such transformations and enumerate all possible values such transformations can assign to a fixed finite simple group. This is done in terms of the first two homology groups. We prove for example that except special linear groups, such an orbit can have at most 7 elements. We also study the action of monomials of idempotent transformations on finite groups and show for example that orbits of this action are always finite.

Keywords: Cellularization of groups; Simple groups

1. Introduction

One way to understand symmetry of some objects is to look for what acts on them and study operations on these objects. In this way we study symmetry of groups by considering endofunctors ϕ : Groups $\to$ Groups. To understand how such an operation ϕ deforms groups we consider natural transformations $\epsilon_G : \phi(G) \to G$. A choice of a functor ϕ : Groups $\to$ Groups and a natural transformation $\epsilon_G : \phi(G) \to G$ is called an augmented functor and denoted by (ϕ, ϵ). By iterating the augmentation we obtain two homomorphisms $\epsilon_{\phi(G)} : \phi^2(G) \to \phi(G)$ and $\phi(\epsilon_G) : \phi^2(G) \to \phi(G)$. Among all augmented functors (ϕ, ϵ) there are the idempotent ones

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for which this iteration process does not produce anything new and the homomorphisms $\epsilon_{\phi(G)}$ and $\phi(\epsilon_G)$ are isomorphisms for any group G. The universal central extension of the maximal perfect subgroup of G, with the natural projection as augmentation, is an example of an idempotent functor. Idempotent functors are related to the concept of cellularity which was introduced originally in homotopy theory and has been used to organize information about spaces. In recent years these functors have been considered in algebraic context of groups, chain complexes, etc.; see for example [1,3–11,18,20,16].

The main aim of this paper is to understand how idempotent functors deform finite groups, particularly the simple ones. Our first result is (see Corollary 4.4, where preservation of nilpotency and solvability is also discussed) the following.

Theorem A. *Let (ϕ, ϵ) be an idempotent functor. If G is finite, then so is $\phi(G)$.*

In this way finite groups are acted upon by idempotent functors. How complicated is this action? To measure it, we study the orbits of this action.

Definition 1.1. $\mathrm{Idem}(G) := \{$isomorphism class of $\phi(G) \mid (\phi, \epsilon)$ is idempotent$\}$.

Although the collection of idempotent functors does not even form a set, the number of different values idempotent functors can take on a given finite group is finite (see Corollary 6.10).

Theorem B. *If G is a finite group, then* $\mathrm{Idem}(G)$ *is a finite set.*

One might then try to enumerate this set. One aim of this paper is to do that for finite simple groups for which we find that $\mathrm{Idem}(G)$ has in general very few elements (see Corollary 8.2 and Section 11). Recall that by functoriality $\mathrm{Aut}(G)$ acts on the Schur multiplier $H_2(G)$ of G. Let $\mathrm{InvSub}(H_2(G))$ denote the set of all subgroups of $H_2(G)$ which are invariant (not necessarily pointwise fixed) under this action.

Theorem C. *Let G be a finite simple group. There is a bijection between* $\mathrm{Idem}(G)$ *and the set:*

$$\{0\} \coprod \mathrm{InvSub}(H_2(G)).$$

The bijection in the above theorem can be described explicitly. The value of an idempotent functor on G could be the trivial group. This corresponds to the element 0 in the above set. If the value is not trivial, then it can be constructed as follows: first take the universal central extension $H_2(G) \lhd E \to G$, then, for an invariant subgroup $K \subset H_2(G)$, take the quotient E/K. Such quotients are exactly the non-trivial values idempotent functors take on a simple group G.

The composition $(\phi'\phi, \epsilon'_{\phi(-)}\epsilon)$ of two idempotent functors (ϕ, ϵ) and (ϕ', ϵ') is not, in general, idempotent. Such compositions give new operations on groups. We can then study the orbits of the action of this broader collection of operations.

Definition 1.2. Let n be a positive integer.

$$\mathrm{Idem}^n(G) := \{\text{isomorphism classes of } \phi_1 \cdots \phi_n(G) \mid \text{for all } i, (\phi_i, \epsilon_i) \text{ is idempotent}\}.$$

Since the identity functor, with the augmentation given by the identity, is idempotent, $\mathrm{Idem}^1(G) \subset \mathrm{Idem}^2(G) \subset \mathrm{Idem}^3(G) \cdots$. Using this increasing sequence of inclusions, we define:

$$\mathrm{Idem}^\infty(G) := \bigcup_{k \geq 1} \mathrm{Idem}^k(G).$$