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Étale splittings of certain Azumaya algebras on toric and hypertoric varieties in positive characteristic

Theodore J. Stadnik Jr.*

Department of Mathematics, Northwestern University, Evanston, IL 60208, United States

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Abstract

For a smooth toric variety X over a field of positive characteristic, a T-equivariant étale cover $Y \to T^*X^{(1)}$ trivializing the sheaf of crystalline differential operators on X is constructed. This trivialization is used to show that $\mathcal{D}$ is a trivial Azumaya algebra along the fibers of the moment map $\mu : T^*X^{(1)} \to \mathfrak{t}^{*(1)}$. This result is then extended to certain Azumaya algebras on hypertoric varieties, whose global sections are analogous to central reductions of the hypertoric enveloping algebra. A criterion for a derived Beilinson–Bernstein localization theorem is then formulated.

Keywords: Azumaya algebra; D-module; Derived localization; Symplectic algebraic geometry

Contents

* Correspondence to: Department of Mathematics, The University of California at Berkeley, Berkeley, CA 94702, United States.
E-mail address: tstadnik@math.berkeley.edu.

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1. Introduction

In the seminal article [17], I. Mussen and M. Van den Bergh study central reductions of the hypertoric enveloping algebra in characteristic zero. The following article will construct these central reductions in positive characteristic as the global sections of a sheaf of Azumaya algebras on the corresponding hypertoric variety. It is shown that this Azumaya algebra is split along the fibers of the natural resolution of singularities $\mathscr{M} \to \mathscr{M}_0$. This gives a hypertoric variant of a result by R. Bezrukavnikov, I. Mirković, and D. Rumynin [2, 5.1.1]. The work of Braden–Licata–Proudfoot–Webster [4] and Bellamy–Kuwabara [1] demonstrates there is a good theory of category $\mathcal{O}$ and localization in characteristic 0. Prompted by this fact, the formalism of [2] is combined with the positive characteristic Grauert–Riemenschneider theorem to prove that a derived Beilinson–Bernstein theorem holds whenever the localization functor has finite homological dimension. In order to accomplish these theorems, it is first necessary to consider the sheaf of differential operators on a smooth toric variety.

The sheaf of crystalline differential operators, $\mathcal{D}$, on a smooth variety $X/\mathbb{F}_p$ has traditionally been studied in number theory. It was classically known that $\mathcal{D}$ is a trivial Azumaya algebra when restricted to the zero section $X^{(1)} \to T^*X^{(1)}$. This fact is extensively used in the study of Grothendieck's sheaf of differential operators $\mathbb{D}$ (See [8]). Until the appearance of [2], this was the most popular application of the theory. In [2], it was shown that $\mathcal{D}$ is an Azumaya algebra over the space $T^*X^{(1)}$ when X is a smooth variety over an algebraically closed field of characteristic p. They also construct a flat (but wildly ramified) cover $T^{*,1}X \to T^*X^{(1)}$ which trivializes $\mathcal{D}$. Based on a choice of a splitting for the Cartier operator, A. Ogus and V. Vologodsky [18] succeeded in trivializing $\mathcal{D}$ on an étale cover. The authors of [2] also discovered that $\mathcal{D}_{G/B}$ restricted to the fibers of the moment map $\mu : T^*G/B^{(1)} \to \mathfrak{g}^{*(1)}$ is trivial for G semi-simple and all p large enough. In the same paper it is also shown there is an equivalence of categories $D^b(U(\mathfrak{g})^\lambda) \cong D^b(\mathcal{D}^\lambda_{G/B})$ for regular weights λ, a derived version of Beilinson–Bernstein localization.

This paper will look to generalize these results in the case of toric and hypertoric varieties, using only linear algebra and the construction of the flat cover. In the case that $T \subset X$ is a toric variety, a T-equivariant étale cover $Y \to T^*X^{(1)}$ which T-equivariantly trivializes $\mathcal{D}$ is constructed. It is based upon the observation that the Euler operator $x\partial_x$ on $\mathbb{A}^1$ satisfies the Artin–Schreier equation (with $\mathcal{O}_{T^*X^{(1)}}$-coefficients) $p(W) = W^p - W - x^p\partial^p$ and is T-invariant. Its relation to the work of Ogus–Vologodsky is discussed in Remark 3.3.6. A slightly different description of Y is used to prove that $\mathcal{D}$ is trivial when restricted to the fibers of the moment map $\mu : T^*X^{(1)} \to \mathfrak{t}^{*(1)}$. This construction is then used to create analogous étale covers splitting