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Advances in Mathematics 233 (2013) 291–331

ADVANCES IN Mathematics

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Generic combinatorial rigidity of periodic frameworks

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Received 15 June 2010; accepted 16 October 2012
Available online 8 November 2012

Communicated by Gil Kalai

Abstract

We give a combinatorial characterization of generic minimal rigidity for *planar periodic frameworks*. The characterization is a true analogue of the Maxwell–Laman Theorem from rigidity theory: it is stated in terms of a finite combinatorial object and the conditions are checkable by polynomial time combinatorial algorithms.

To prove our rigidity theorem we introduce and develop *periodic direction networks* and $\mathbb{Z}^2$*-graded-sparse colored graphs*.

Keywords: Combinatorial rigidity; Matroids; Periodic graphs

1. Introduction

A *periodic framework* is an infinite planar structure, periodic with respect to a lattice representing $\mathbb{Z}^2$, made of *fixed-length bars* connected by joints with full rotational degrees of freedom; the allowed continuous motions are those that preserve the lengths and connectivity of the bars, and the framework's $\mathbb{Z}^2$-symmetry. A periodic framework is *rigid* if the only allowed motions are Euclidean isometries, and flexible otherwise.

The forced periodicity is a key feature of this model: there are structures that are rigid with respect to periodicity-preserving motions that are flexible if a larger class of motions is allowed. What is *not* required to be preserved is also noteworthy: the lattice is allowed to change as the framework moves.

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.10.007

Formally, a periodic framework is given by a triple $(\tilde{G}, \varphi, \tilde{\boldsymbol{\ell}})$ where: $\tilde{G}$ is a simple infinite graph; φ is a free $\mathbb{Z}^2$-action on $\tilde{G}$ by automorphisms such that the quotient is finite; and $\tilde{\boldsymbol{\ell}} = (\tilde{\ell_{ij}})$ assigns a length to each edge of $\tilde{G}$.

A *realization* $\tilde{G}(\mathbf{p}, \mathbf{L})$ of a periodic framework $(\tilde{G}, \varphi, \tilde{\boldsymbol{\ell}})$ is defined to be a mapping $\mathbf{p}$ of the vertex set $V(\tilde{G})$ into $\mathbb{R}^2$ and a representation $\mathbb{Z}^2 \to \mathbb{R}^2$ encoded by a matrix $\mathbf{L} \in \mathbb{R}^{2\times 2}$ (with $\mathbb{R}^2$ here viewed as translations) such that:

- the representation is equivariant with respect to the $\mathbb{Z}^2$-actions on $\tilde{G}$ and the plane; i.e., $\mathbf{p}_{\gamma\cdot i} = \mathbf{p}_i + \mathbf{L}\cdot\gamma$ for all $i \in V(\tilde{G})$ and $\gamma \in \mathbb{Z}^2$.
- The specified edge lengths are preserved by $\mathbf{p}$; i.e., $\|\mathbf{p}_i - \mathbf{p}_j\| = \tilde{\ell}_{ij}$ for all edges $ij \in E(\tilde{G})$.

The reader should note that together these definitions imply that, to be realizable, an abstract periodic framework must give the same length to each $\mathbb{Z}^2$-orbit of edges.

A realization $\tilde{G}(\mathbf{p}, \mathbf{L})$ is *rigid* if the only allowed continuous motions of $\mathbf{p}$ and $\mathbf{L}$ that preserve the action φ and the edge lengths are rigid motions of the plane and *flexible* otherwise. If $\tilde{G}(\mathbf{p}, \mathbf{L})$ is rigid but ceases to be so if any $\mathbb{Z}^2$-orbit of edges in $\tilde{G}$ is removed it is minimally rigid. These definitions of periodic frameworks and rigidity are from [2]. (See Section 16 for complete details.)

1.1. Main theorem

The topic of this paper is to determine rigidity and flexibility of periodic frameworks based only on the *combinatorics* of a framework—i.e., which bars are present and not their specific lengths. In general, this is not possible, and even testing rigidity of a *finite* framework seems to be a hard problem, with the best known algorithms relying on exponential-time Gröbner basis computations.

However, for *generic* periodic frameworks, we give the following combinatorial characterization, which is analogous to the landmark Maxwell–Laman Theorem [14,8]. The *colored-Laman graphs* appearing in statements of theorems are defined in Section 4; the quotient graph is defined in Section 2, and genericity is defined precisely in Section 17 in terms of the coordinates of the points in a realization avoiding a nowhere-dense algebraic set. In particular, this means that the set non-generic realizations has measure zero.

Theorem A. *Let $(\tilde{G}, \varphi, \tilde{\boldsymbol{\ell}})$ be a generic periodic framework. Then a generic realization $\tilde{G}(\mathbf{p}, \mathbf{L})$ of $(\tilde{G}, \varphi, \tilde{\boldsymbol{\ell}})$ is minimally rigid if and only if its colored quotient graph $(G, \boldsymbol{\gamma})$ is colored-Laman.*

Theorem A is a true combinatorial characterization of generic periodic rigidity in the plane: $(G, \boldsymbol{\gamma})$ is a finite combinatorial object and the colored-Laman condition is checkable in polynomial time. The specialization of Theorem A to the case where the quotient graph has only one vertex is implied by [2, Theorem 3.12].

1.2. Examples

Because infinite periodic graphs are unwieldy to work with, we will model periodic frameworks by *colored graphs*, which are finite directed graphs with elements of $\mathbb{Z}^2$ on the edges. These are defined in Section 2, but we show some examples here to give intuition and motivate Theorem A. Fig. 1(a) shows part of a periodic point set, (b) makes it more clear that it is indeed periodic by indicating the $\mathbb{Z}^2$ orbits of the points; (c) indicates the vectors representing $\mathbb{Z}^2$ by translations and shows several copies of the fundamental domain of the $\mathbb{Z}^2$-action on