Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 235 (2013) 1–55

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Traces of compact operators and the noncommutative residue

Nigel Kalton [1], Steven Lord [a], Denis Potapov [b], Fedor Sukochev [b,*]

[a] *School of Mathematical Sciences, University of Adelaide, Adelaide, 5005, Australia*
[b] *School of Mathematics and Statistics, University of New South Wales, Sydney, 2052, Australia*

Received 21 February 2012; accepted 20 November 2012
Available online 20 December 2012

Communicated by Alain Connes

Abstract

We extend the noncommutative residue of M. Wodzicki on compactly supported classical pseudo-differential operators of order $-d$ and generalise A. Connes' trace theorem, which states that the residue can be calculated using a singular trace on compact operators. Contrary to the role of the noncommutative residue for the classical pseudo-differential operators, a corollary is that the pseudo-differential operators of order $-d$ do not have a 'unique' trace; pseudo-differential operators can be non-measurable in Connes' sense. Other corollaries are given clarifying the role of Dixmier traces in noncommutative geometry, including the definitive statement of Connes' original theorem.

MSC: 47B10; 58B34; 58J42; 47G10

Keywords: Noncommutative residue; Connes' trace theorem; Lidskii theorem; Noncommutative geometry; Spectral theory; Singular trace

1. Introduction

A. Connes proved, [8, Theorem 1], that

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\mathrm{Res}_W(P)$$

* Corresponding author.
E-mail addresses: steven.lord@adelaide.edu.au (S. Lord), d.potapov@unsw.edu.au (D. Potapov), f.sukochev@unsw.edu.au (F. Sukochev).

[1] Nigel Kalton (1946–2010). The author passed away during the production of this paper.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.11.007

where P is a classical pseudo-differential operator of order $-d$ on a d-dimensional closed Riemannian manifold, Tr_ω is a Dixmier trace (a trace on the compact operators with singular values $O(n^{-1})$ which is not an extension of the canonical trace), [13], and Res_W is the noncommutative residue of M. Wodzicki, [44].

Connes' trace theorem, as it is known, has become the cornerstone of noncommutative integration in noncommutative geometry, [9]. Applications of Dixmier traces as the substitute noncommutative residue and integral in non-classical spaces range from fractals, [31,22], to foliations [3], to spaces of noncommuting co-ordinates, [10,21], and applications in string theory and Yang–Mills, [12,14,40,8], Einstein–Hilbert actions and particle physics' standard model, [11,6,30].

Connes' trace theorem, though, is not complete. There are other traces, besides Dixmier traces, on the ideal of compact operators whose singular values are $O(n^{-1})$. Wodzicki showed that the noncommutative residue is essentially the unique trace on classical pseudo-differential operators of order $-d$, so it should be expected that every suitably normalised trace computes the noncommutative residue. Also, all pseudo-differential operators have a notion of principal symbol and Connes' trace theorem opens the question of whether the principal symbol of non-classical operators can be used to compute their Dixmier trace.

We generalise Connes' trace theorem. We introduce an extension of the noncommutative residue that relies only on the principal symbol of a pseudo-differential operator, and we show that the extension calculates the Dixmier trace of the operator. The following definition and theorem apply to a much wider class of Hilbert–Schmidt operators, called Laplacian modulated operators, that we develop in the text. Here, in the introduction, we mention only pseudo-differential operators.

A pseudo-differential operator $P : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ is compactly based if Pu has compact support for all $u \in C^\infty(\mathbb{R}^d)$. Equivalently the (total) symbol of P has compact support in the first variable.

Definition 1.1 (*Extension of the Noncommutative Residue*). Let $P : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ be a compactly based pseudo-differential operator of order $-d$ with symbol p. The linear map

$$P \mapsto \mathrm{Res}(P) := \left[\left\{ \frac{d}{\log(1+n)} \int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p(x,\xi) d\xi \, dx \right\}_{n=1}^{\infty} \right]$$

we call the *residue* of P, where $[\cdot]$ denotes the equivalence class in ℓ_∞ / c_0.

Here ℓ_∞ denotes the space of bounded complex-valued sequences, and c_0 denotes the subspace of vanishing at infinity convergent sequences. Alternatively, any sequence $\mathrm{Res}_n(P)$, $n \in \mathbb{N}$, such that

$$\int_{\mathbb{R}^d} \int_{|\xi| \le n^{1/d}} p(x,\xi) d\xi \, dx = \frac{1}{d} \mathrm{Res}_n(P) \log n + o(\log n)$$

defines the residue $\mathrm{Res}(P) = [\mathrm{Res}_n(P)] \in \ell_\infty / c_0$. We identify the equivalence classes of constant sequences in ℓ_∞ / c_0 with scalars. In the case that $\mathrm{Res}(P)$ is the class of a constant sequence, then we say that $\mathrm{Res}(P)$ is a scalar and identify it with the limit of the constant sequence. Note that a dilation invariant state $\omega \in \ell_\infty^*$ vanishes on c_0. Hence

$$\omega([c_n]) := \omega(\{c_n\}_{n=1}^\infty), \quad \{c_n\}_{n=1}^\infty \in \ell_\infty$$

is well-defined as a linear functional on ℓ_∞ / c_0.