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Semiregularity and obstructions of complete intersections

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Abstract

We prove that, on a smooth projective variety over an algebraically closed field of characteristic 0, the semiregularity map annihilates every obstruction to embedded deformations of a local complete intersection subvariety with extendable normal bundle. The proof is based on the theory of L_∞-algebras and Tamarkin–Tsygan calculus on the de Rham complex of DG-schemes.

Keywords: Deformation theory; DG-schemes; Obstruction theory; Differential graded Lie algebras; Local cohomology

0. Introduction

Let X be a smooth algebraic variety, over an algebraically closed field $\mathbb{K}$ of characteristic 0, and let $Z \subset X$ be a locally complete intersection closed subvariety of codimension p. Following [3], the *semiregularity map* $\pi\colon H^1(Z, N_{Z|X}) \to H^{p+1}(X, \Omega_X^{p-1})$, where $N_{Z|X}$ is the normal bundle of Z in X, can be conveniently described by using local cohomology. In fact, since Z is a locally complete intersection, it is well defined the *canonical*

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cycle class $\{Z\}' \in \Gamma(X, \mathcal{H}^p_Z(\Omega^p_X))$ [3, p. 59] and the contraction with it gives a morphism of sheaves

$$N_{Z|X} \xrightarrow{\lrcorner\{Z\}'} \mathcal{H}^p_Z(\Omega^{p-1}_X).$$

Passing to cohomology, we get a map

$$H^1(Z, N_{Z|X}) \xrightarrow{\lrcorner\{Z\}'} H^1(Z, \mathcal{H}^p_Z(\Omega^{p-1}_X)) = H^{p+1}_Z(X, \Omega^{p-1}_X),$$

where the last equality follows from the spectral sequence of local cohomology. Then, the semiregularity map is obtained by taking the composition with the natural map $H^{p+1}_Z(X, \Omega^{p-1}_X) \to H^{p+1}(X, \Omega^{p-1}_X)$.

In [3], using Hodge theory and de Rham cohomology, S. Bloch proved that if X is projective, then the semiregularity map annihilates certain obstructions to embedded deformations of Z in X. These obstructions contain in particular the curvilinear ones and, therefore, if the semiregularity map is injective, then the Hilbert scheme of subschemes of X is smooth at Z.

Unfortunately, Bloch's argument is not sufficient to ensure that the semiregularity map annihilates every obstruction to deformations. We have two main reasons to extend the Bloch theorem to every obstruction: the first is for testing the power of *derived deformation theory* in a problem where classical deformation theory has failed for almost 40 years. This new approach already worked when Z is a smooth submanifold, see [38,23,25] and Remark 11.2, and the solution of this particular case has given a deep insight about the most appropriate formulation and more useful tools of derived deformation theory. The second reason is related with the theory of reduced Gromov–Witten invariants. Indeed, to define the GW invariants, one needs the virtual fundamental class, defined through an obstruction theory. Whenever the obstruction theory is not carefully chosen, then the virtual fundamental class is zero and the standard GW theory is trivial. A way to overcome this problem, and perform a non trivial Gromov–Witten theory is by using a reduced obstruction theory, obtained by considering the kernel of a suitable map annihilating obstructions [34,31].

The philosophy of derived deformation theory may be summarized in the following way (see e.g. [35]): over a field of characteristic 0, every deformation problem is the classical truncation of an extended deformation problem, which is controlled by a differential graded Lie algebra via the Maurer–Cartan equation and gauge equivalence. This differential graded Lie algebra is defined up to quasi-isomorphism and its first cohomology group is equal to the Zariski tangent space of the local moduli space. A morphism of deformation theories is essentially a morphism in the derived category of differential graded Lie algebras (with quasi-isomorphisms as weak equivalences); the induced morphism in cohomology gives, in degrees 1 and 2, the tangent and obstruction map, respectively.

Clearly, a morphism from a deformation theory into an unobstructed deformation theory provides an obstruction map annihilating every obstruction. Using this basic principle, we are able to prove that the semiregularity map annihilates every obstruction under the following additional assumption.

Set-up: *Z is closed of codimension p in X and there exists a Zariski open subset $U \subset X$ and a vector bundle $E \to U$ of rank p such that $Z \subset U$ and Z is the zero locus of a section $f \in \Gamma(U, E)$.*

This is obviously satisfied for complete intersections of hypersurfaces, while for Z of codimension 2 we refer to [41] for a discussion about the validity of the above set-up.