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Boundedness of extremal solutions in dimension 4

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Abstract

In this paper we establish the boundedness of the extremal solution u^* in dimension $N = 4$ of the semilinear elliptic equation $-\Delta u = \lambda f(u)$, in a general smooth bounded domain $\Omega \subset \mathbb{R}^N$, with Dirichlet data $u|_{\partial\Omega} = 0$, where f is a C^1 positive, nondecreasing and convex function in $[0, \infty)$ such that $f(s)/s \to \infty$ as $s \to \infty$.

In addition, we prove that, for $N \geq 5$, the extremal solution $u^* \in W^{2,\frac{N}{N-2}}$. This gives $u^* \in L^{\frac{N}{N-4}}$, if $N \geq 5$ and $u^* \in H_0^1$, if $N = 6$.

Keywords: Extremal solutions; Boundedness; Elliptic equations

1. Introduction and main results

In this paper, we consider the following semilinear elliptic equation, which has been extensively studied:

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1, \lambda \geq 0$ is a real parameter and the nonlinearity $f : [0, \infty) \to \mathbb{R}$ satisfies

$$f \text{ is } C^1, \text{ nondecreasing and convex,} \qquad f(0) > 0, \quad \text{and} \quad \lim_{u\to+\infty} \frac{f(u)}{u} = +\infty. \qquad (1.1)$$

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It is well known that there exists a finite positive extremal parameter λ^* such that (P_λ) has a minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ if $0 \leq \lambda < \lambda^*$, while no solution exists, even in the weak sense, for $\lambda > \lambda^*$. The set $\{u_\lambda : 0 \leq \lambda < \lambda^*\}$ forms a branch of classical solutions increasing in λ. Its increasing pointwise limit $u^*(x) := \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ is a weak solution of (P_λ) for $\lambda = \lambda^*$, which is called the extremal solution of (P_λ) (see [2,3,9]). In fact, if f satisfies all the hypotheses of (1.1) except the convexity, then all the results we have mentioned remain true, except the continuity of the family of minimal solutions $\{u_\lambda\}$ as a function of λ (see [5, Proposition 5.1]).

The regularity and properties of extremal solutions depend strongly on the dimension N, domain Ω and nonlinearity f. When $f(u) = e^u$, it is known that $u^* \in L^\infty(\Omega)$ if $N < 10$ (for every Ω) (see [8,11]), while $u^*(x) = -2 \log |x|$ and $\lambda^* = 2(N-2)$ if $N \geq 10$ and $\Omega = B_1$ (see [10]). There is an analogous result for $f(u) = (1+u)^p$ with $p > 1$ (see [3]). Brezis and Vázquez [3] raised the question of determining the boundedness of u^*, depending on the dimension N, for general nonlinearities f satisfying (1.1). The first general results were due to Nedev [12], who proved that $u^* \in L^\infty(\Omega)$ if $N \leq 3$, and $u^* \in L^p(\Omega)$ for every $p < N/(N-4)$, if $N \geq 4$. The best known result was established by Cabré [4], who proved that $u^* \in L^\infty(\Omega)$ if $N \leq 4$ and Ω is convex (no convexity on f is imposed). If $N \geq 5$ and Ω is convex Cabré and Sanchón [7] have obtained that $u^* \in L^{\frac{2N}{N-4}}(\Omega)$ (again, no convexity on f is imposed). On the other hand, Cabré and Capella [5] have proved that $u^* \in L^\infty(\Omega)$ if $N \leq 9$ and $\Omega = B_1$. Recently, Cabré and Ros-Oton [6] have obtained that $u^* \in L^\infty(\Omega)$ if $N \leq 7$ and Ω is a convex domain of double revolution (see [6] for the definition).

Another interesting question is whether the extremal solution lies in the energy class. Nedev [12,13] proved that $u^* \in H_0^1(\Omega)$ if $N \leq 5$ (for every Ω) or Ω is convex (for every $N \geq 1$). Brezis and Vázquez [3] proved that a sufficient condition to have $u^* \in H_0^1(\Omega)$ is that $\liminf_{u \to \infty} u f'(u)/f(u) > 1$ (for every Ω and $N \geq 1$).

In this paper we establish the boundedness of the extremal solution for general bounded smooth domains in dimension 4, not necessarily convex. Contrary to the result of Cabré, we need to impose the convexity of f. In higher dimensions, we improve the results of Nedev [12,13] and it is obtained that $u^* \in L^{\frac{N}{N-4}}(\Omega)$, if $N \geq 5$ and $u^* \in H_0^1(\Omega)$, if $N = 6$.

Theorem 1.1. *Let f be a function satisfying (1.1) and $\Omega \subset \mathbb{R}^4$ be a smooth bounded domain. Let u^* be the extremal solution of (P_λ). Then $u^* \in L^\infty(\Omega)$.*

Theorem 1.2. *Let f be a function satisfying (1.1) and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let u^* be the extremal solution of (P_λ). Then, for $N \geq 5$, $u^* \in W^{2,\frac{N}{N-2}}(\Omega)$ and $f(u^*) \in L^{\frac{N}{N-2}}(\Omega)$. In particular,*

(i) *If $N \geq 5$, then $u^* \in L^{\frac{N}{N-4}}(\Omega)$.*
(ii) *If $N = 6$, then $u^* \in H_0^1(\Omega)$.*

The proofs of Theorems 1.1 and 1.2 use the semi-stability of the minimal solutions u_λ ($0 < \lambda < \lambda^*$).

Recall that a classical solution u of

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$