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New enumeration formulas for alternating sign matrices and square ice partition functions

Arvind Ayyer, Dan Romik*

Department of Mathematics, University of California, Davis, One Shields Ave., Davis, CA 95616, USA

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Abstract

The refined enumeration of alternating sign matrices (ASMs) of given order having prescribed behavior near one or more of their boundary edges has been the subject of extensive study, starting with the Refined Alternating Sign Matrix Conjecture of Mills–Robbins–Rumsey (1983) [25], its proof by Zeilberger (1996) [31], and more recent work on doubly-refined and triply-refined enumerations by several authors. In this paper we extend the previously known results on this problem by deriving explicit enumeration formulas for the "top–left–bottom" (triply-refined) and "top–left–bottom–right" (quadruply-refined) enumerations. The latter case solves the problem of computing the full boundary correlation function for ASMs. The enumeration formulas are proved by deriving new representations, which are of independent interest, for the partition function of the square ice model with domain wall boundary conditions at the "combinatorial point" $\eta = 2\pi/3$.

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* Corresponding author.
E-mail addresses: ayyer@math.ucdavis.edu (A. Ayyer), romik@math.ucdavis.edu (D. Romik).

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$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Fig. 1. An alternating sign matrix of order 6.

1. Introduction

1.1. Alternating sign matrices

An *alternating sign matrix (ASM)* of order n is an $n \times n$ square matrix with entries in $\{0, 1, -1\}$ such that in every row and every column, the sum of the entries is 1 and the non-zero terms appear with alternating signs; see Fig. 1 for an example. From this seemingly innocuous definition a uniquely fascinating class of objects arises: originally discovered by Mills, Robbins and Rumsey in connection with their study of Dodgson's condensation method for computing determinants, ASMs have since been found to have deep connections to many other topics of interest in combinatorics and statistical physics. Some places where ASMs make an unexpected appearance are: the *square ice model* (a.k.a. the *six-vertex model*) [23,24,29,31], *totally symmetric self complementary plane partitions* [20,26], *descending plane partitions* [25], *domino tilings* [15], and the $O(1)$ *loop model* in a cylindrical geometry [9,28].

1.2. Enumeration of alternating sign matrices

The current paper will be focused on one particular aspect of the study of ASMs, namely the problem of enumerating various naturally-occurring sets of ASMs of some fixed order n. This is a problem with a venerable history, starting with the seminal paper [25] of Mills, Robbins and Rumsey. Having defined ASMs and discovered the role that they play in the definition of the λ-*determinant*, a natural generalization of matrix determinants, Mills et al. considered the problem of finding the total number A_n of ASMs of order n. Based on numerical observations, they conjectured the formula

$$A_n = \frac{1!\,4!\,7! \cdots (3n-2)!}{n!(n+1)! \cdots (2n-1)!} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \prod_{j=0}^{n-1} \frac{\binom{3j+1}{j}}{\binom{2j}{j}}. \tag{1.1}$$

Another natural enumeration problem concerned the so-called *refined* enumeration of ASMs. It is based on the trivial observation (an immediate consequence of the definition of ASMs) that the top row of an ASM contains a single 1 and no -1s. The position of the 1 in the top row is therefore an interesting parameter by which one may refine the total enumeration. Thus, for $1 \le k \le n$ Mills et al. defined

$$A_{n,k} = \#\text{ of ASMs of order } n \text{ with 1 in position } (1,k),$$

and conjectured that

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!} \quad (1 \le k \le n). \tag{1.2}$$

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