Available online at www.sciencedirect.com

SciVerse ScienceDirect

ADVANCES IN Mathematics

Advances in Mathematics 235 (2013) 361–389

www.elsevier.com/locate/aim

Conjugacy growth of finitely generated groups

Michael Hull, Denis Osin*

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

Received 3 December 2011; accepted 11 December 2012
Available online 7 January 2013

Communicated by Matthew Ando

Abstract

The conjugacy growth function of a finitely generated group measures the number of conjugacy classes in balls with respect to a word metric. We study the following natural question: Which functions can occur as the conjugacy growth function of finitely generated groups? Our main result answers the question completely. Namely we prove that a function $f : \mathbb{N} \to \mathbb{N}$ can be realized (up to a natural equivalence) as the conjugacy growth function of a finitely generated group if and only if f is non-decreasing and bounded from above by a^n for some $a \geq 1$. We also construct a finitely generated group G and a subgroup $H \leq G$ of index 2 such that H has only 2 conjugacy classes while the conjugacy growth of G is exponential. In particular, conjugacy growth is not a quasi-isometry invariant.

MSC: 20F69; 20E45; 20F65; 20F67

Keywords: Conjugacy growth function; Small cancellation theory; Relatively hyperbolic group; Quasi-isometry

1. Introduction

Let G be a group generated by a set X. Recall that the *word length* of an element $g \in G$ with respect to the generating set X, denoted by $|g|_X$, is the length of a shortest word in $X \cup X^{-1}$ representing g in the group G. If X is finite one can consider the *growth function* of G, $\gamma_G : \mathbb{N} \to \mathbb{N}$, defined by

$$\gamma_G(n) = |B_{G,X}(n)|,$$

* Corresponding author.
E-mail addresses: michael.b.hull@vanderbilt.edu (M. Hull), denis.osin@gmail.com (D. Osin).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.12.007

where

$$B_{G,X}(n) = \{g \in G \mid |g|_X \leq n\}.$$

It was first introduced by Efremovic [17] and Svarč [43] in the 50's, rediscovered by Milnor [34] in the 60's, and served as the starting point and a source of motivating examples for contemporary geometric group theory. In this paper we focus on a similar function $\xi_{G,X} : \mathbb{N} \to \mathbb{N}$ called the *conjugacy growth function* of G with respect to X. By definition $\xi_{G,X}(n)$ is the number of conjugacy classes in the ball $B_{G,X}(n)$.

It is straightforward to verify that $\gamma_{G,X}$ and $\xi_{G,X}$ are independent of the choice of a particular finite generating set X of G up to the following equivalence relation. Given $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there exists $C \in \mathbb{N}$ such that $f(n) \leq g(Cn)$ for all $n \in \mathbb{N}$. Further f and g are *equivalent* (we write $f \sim g$) if $f \preceq g$ and $g \preceq f$. In what follows we always consider growth functions up to this equivalence relation and omit X from the notation.

The conjugacy growth function was introduced by Babenko [2] in order to study geodesic growth of Riemannian manifolds. Obviously free homotopy classes of loops in a manifold M are in 1-to-1 correspondence with conjugacy classes of $\pi_1(M)$. If M is a closed Riemannian manifold, the proposition known as the Svarč–Milnor Lemma (and first proved by Efremovic in [17]) then implies that $\xi_{\pi_1(M)}$ is equivalent to the function counting free homotopy classes of loops of given length in M. The later function serves as a lower bound for the geodesic growth function of M, which counts the number of geometrically distinct closed geodesics of given length on M. Moreover if M has negative sectional curvature, then all these functions are equivalent.

Geodesic growth of compact Riemannian manifolds has been studied extensively since late 60's (see, e.g., [3,4,30,32]). The most successful results were obtained in the case of negatively curved manifolds by Margulis [32,33]. He proved that the number of primitive closed geodesics of length at most n on a closed manifold of negative sectional curvature is approximately equal to $e^{hn}/(hn)$, where h is the topological entropy of the geodesic flow on the unit tangent bundle of the manifold. Coornaert and Knieper [11,12] proved a group theoretic analogue of this result and found an asymptotic estimate for the number of primitive conjugacy classes in a hyperbolic group similar to that from Margulis' papers.

Recall that a conjugacy class of a group G is called *primitive* if some (or, equivalently, any) element g from the class is not a proper power, i.e., $h^n = g$ implies $n = \pm 1$. For a group G generated by a finite set X, let $\pi_G(n)$ denote the function counting primitive conjugacy classes in $B_{G,X}(n)$. It is not hard to show that π_G and ξ_G are equivalent and grow exponentially for many 'hyperbolic-like' groups. Indeed we prove the following.

Theorem 1.1 (*Theorem 3.6*)**.** *Let G be a finitely generated group with a non-degenerate hyperbolically embedded subgroup. Then $\xi_G \sim \pi_G \sim 2^n$.*

The notion of a hyperbolically embedded subgroup was introduced in [14]. The condition that the subgroup is non-degenerate simply means that it is proper and infinite. Groups containing non-degenerate hyperbolically embedded subgroups include non-elementary hyperbolic and, more generally, relatively hyperbolic groups with proper parabolic subgroups, all but finitely many mapping class groups of closed orientable surfaces (possibly with punctures), $Out(F_n)$ for $n \geq 2$, the Cremona group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ (i.e., the group of birational automorphism of the complex projective plane), and many other examples [14]. Theorem 1.1 can be used to completely classify conjugacy growth functions of subgroups in mapping class groups (see Section 3).