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Every point is critical

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Abstract

We show that, for any compact Alexandrov surface S (without boundary) and any point y in S, there exists a point x in S for which y is a critical point. Moreover, we prove that uniqueness characterizes the surfaces homeomorphic to the sphere among smooth orientable surfaces.

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1. Introduction

In this paper, by *surface* we always mean a compact 2-dimensional Alexandrov space with curvature bounded below and without boundary, as defined by Burago, Gromov and Perelman in [1]. It is known that our surfaces are topological manifolds (see [1, Section 11]). Let $\mathcal{A}$ be the space of all surfaces.

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For any surface S, denote by ρ its metric, and by ρ_x the distance function from x, given by $\rho_x(y) = \rho(x, y)$. A point $y \in S$ is called *critical* with respect to ρ_x (or to x), if for any direction τ of S at y there exists a *segment* (i.e., a shortest path) from y to x whose direction at y makes an angle not greater than $\pi/2$ with τ. For the definition of the set of directions at an arbitrary point of an Alexandrov surface, see again [1]. A *geodesic* is a curve which is locally a segment. We recall that geodesics of S do not bifurcate [1].

The survey [2] by K. Grove presents the principles, as well as applications, of the critical point theory for distance functions.

Every point on a surface admits a critical point. It suffices, indeed, to take a point farthest from it. Conversely, is it true that every point is a critical point of some other point? Certainly, not every point on every surface is a farthest point from some other point: On an ellipsoid of revolution with an NS-axis much longer than the other two, no point of the equator is farthest from any other point. Concerning the set of all critical points, however, the answer is affirmative, as Theorem 1 shows.

For the set-valued function associating to each point of a surface the set of all farthest points on the surface, the relationship between being single-valued and being surjective is investigated in [11].

Theorem 2 characterizes the smooth orientable surfaces homeomorphic to the sphere.

For any point x in S, denote by Q_x the set of all critical points with respect to x, and by Q_x^{-1} the set of all points $y \in S$ with $x \in Q_y$. Let M_x, F_x be the sets of all relative, respectively absolute, maxima of ρ_x. For properties of Q_x and its subsets M_x and F_x in Alexandrov spaces, see [3,11], and the survey [9].

A forthcoming paper, [4], will provide for orientable surfaces an upper bound for $\mathrm{card} Q_y^{-1}$ depending on the genus, and use it to estimate the cardinality of diametrally opposite sets on S. The case of points y in orientable Alexandrov surfaces, which are common maxima of several distance functions, is treated in [10].

We denote by T_y the space of directions at $y \in S$; the length λT_y of T_y satisfies $\lambda T_y \leq 2\pi$ [1]. If $\lambda T_y < 2\pi$ then y is called a *conical point* of S. A surface without conical points is called *smooth.*

There might exist a direction $\tau \in T_y$ such that no segment starts at y in direction τ. On most convex surfaces, the set of such directions τ, called *singular*, is even residual in T_y, for each y (see Theorem 2 in [12]). However, the set of non-singular directions is always dense in T_y. For those τ, for which there is a geodesic Γ with direction τ at y, a so-called *cut point* $c(\tau)$ is associated, defined by the requirement that the arc $yc(\tau) \subset \Gamma$ is a segment which cannot be extended further beyond $c(\tau)$ (remaining a segment). This is well-defined, because in an Alexandrov space of curvature bounded below segments (and geodesics) do not bifurcate. The set of cut points in all non-singular directions at y is the *cut locus* $C(y)$ of the point y.

Recall that a *tree* in S is a set $T \subset S$ any two points of which can be joined by a unique Jordan arc included in T. A set $L \subset S$ is a *local tree* if each of its points x has a neighbourhood V in S such that the connected component $K_x(V)$ of $L \cap V$ containing x is a tree. The *degree* of a point x of a local tree is the cardinality of the set of components of $K_x(V) \setminus \{x\}$ if the neighbourhood V of x is chosen such that $K_x(V)$ be a tree. A point of the local tree L is called an *extremity* of L if it has degree 1, and a *ramification point* of L if it has degree at least 3.

It is known that $C(y)$, if it is not a single point, is a local tree (see [8, Theorem A, p. 534]), even a tree if S is homeomorphic to the sphere (which is easily seen). Theorem 4 in [14] and Theorem 1 in [12] yield the existence of surfaces S on which the set of all extremities of any cut