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Ellipsoidal billiards in pseudo-Euclidean spaces and relativistic quadrics☆

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Abstract

We study the geometry of confocal quadrics in pseudo-Euclidean spaces of arbitrary dimension d and any signature, and related billiard dynamics. The goal is to give a complete description of periodic billiard trajectories within ellipsoids. The novelty of our approach is the introduction of a new discrete combinatorial geometric structure associated with a confocal pencil of quadrics, a colouring in d colours. This is used to decompose quadrics of $d+1$ geometric types of a pencil into new relativistic quadrics of d relativistic types. A study of what we term discriminant sets of tropical lines Σ^+ and Σ^- and their singularities provides insight into the related geometry and combinatorics. This yields an analytic criterion describing all periodic billiard trajectories, including light-like trajectories as a case of special interest.

Keywords: Confocal quadrics; Poncelet theorem; Periodic billiard trajectories; Minkowski space; Light-like billiard trajectories; Tropic curves

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Contents

1. Introduction

Pseudo-Euclidean spaces and pseudo-Riemannian manifolds occupy a very important position in science as a geometric background for general relativity. A modern account of the mathematical aspects of the theory of relativity was recently published [8]. From a mathematical point of view, in comparison with Euclidean and Riemannian cases, apart from a natural similarity that includes some rather technical adjustments, there are some cases in which a pseudo-Euclidean setting creates new situations and challenging problems. The aim of this paper is to report on such cases in a study of the geometry of confocal quadrics and related billiard dynamics in pseudo-Euclidean spaces.

Recall that in Euclidean d-dimensional space, a general family of confocal quadrics contains exactly d geometric types of non-degenerate quadrics. Moreover, each point is the intersection of d quadrics of different types. Together with some other properties (E1–E5 in Section 4), these facts are crucial for the introduction of Jacobi coordinates and for applications in the theory of separable systems, including billiards. The case of d-dimensional pseudo-Euclidean space highlights a striking difference, since a confocal family of quadrics has $d+1$ geometric types of quadrics. In addition, quadrics of the same type have a nonempty intersection. These seem to be impregnable obstacles to the extension of methods for applying Jacobi type coordinates from the Euclidean case [9] to pseudo-Euclidean spaces.

To overcome this crucial problem, we have created a new feature of confocal pencils of geometric quadrics in pseudo-Euclidean d-dimensional spaces. The novelty of our approach is the introduction of a new discrete, combinatorial-geometric structure associated with a confocal pencil, a colouring in d colours, that transforms a geometric quadric from a pencil into a union of several *relativistic quadrics*. These relativistic quadrics satisfy properties PE1–PE5, analogues of