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Two-dimensional moduli spaces of vector bundles over Kodaira surfaces

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Abstract

We prove that any two-dimensional moduli space of stable 2-vector bundles, in the non-filtrable range, on a primary Kodaira surface is a primary Kodaira surface. If a universal bundle exists, then the two surfaces are homeomorphic up to unramified covers.

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1. Introduction

Non-trivial examples of holomorphic symplectic manifolds have been obtained from moduli spaces of semi-stable sheaves over projective holomorphic symplectic surfaces (Mukai, Tyurin, O'Grady, see [12] and the references therein). By classification, any such surface is either K3

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or abelian. Absent the projectivity assumption, or even the Kähler condition, one more class of holomorphic symplectic surfaces appears. Precisely, these are the primary Kodaira surfaces, which are defined as topologically non-trivial principal elliptic bundles $X \to B$ over an elliptic curve B. It is known that moduli spaces of stable sheaves on primary Kodaira surfaces inherit the holomorphic symplectic structure. In some situations, these moduli spaces are compact [21].

In this paper, we study two-dimensional moduli spaces $\mathcal{M}$ of stable rank-2 vector bundles with fixed determinant and fixed second Chern class on primary Kodaira surfaces; here by "fixed determinant" we mean that the determinant line bundle has a fixed isomorphism type. The first observation is that they are compact if we place ourselves in the *non-filtrable range* (see Section 4); note that this hypothesis is equivalent to the apparently weaker condition (*) from [21] which says that every semi-stable vector bundle with suitable topological invariants is stable. We show that, for these spaces, quasi-universal sheaves exist and that they are sometimes even universal sheaves. Using the spectral cover construction from [7], we infer that the spaces $\mathcal{M}$ are principal elliptic bundles over the same elliptic curve B, hence they are either tori or primary Kodaira surfaces. We perform an analysis of the torus case, and we note that a universal family exists after passing to an étale cover. We are led to a situation where X may be seen as a parameter space for vector bundles on $\mathcal{M}$. Using the techniques developed in [19], we get a contradiction of the fact that $\mathcal{M}$ was supposed to be Kähler. The bottom line is that these spaces $\mathcal{M}$ must be primary Kodaira surfaces.

In the last section, we prove that if a universal sheaf exists, then the moduli space $\mathcal{M}$ is homeomorphic to an unramified cover of the original base surface X.

2. Rank-2 vector bundles on Kodaira surfaces

For any smooth compact complex surface X and any pair of Chern classes $(c_1, c_2) \in \mathrm{NS}(X) \times \mathbb{Z}$, one defines the discriminant as

$$\Delta(2, c_1, c_2) := \frac{1}{2}\left(c_2 - \frac{1}{4}c_1^2\right).$$

On non-algebraic surfaces, the intersection form on the Neron–Severi group is negative semi-definite, and this can be used to show that if (c_1, c_2) are the Chern classes of a rank-2 holomorphic vector bundle on X, then $\Delta(2, c_1, c_2) \geq 0$ [3,6]. A natural problem is to determine whether or not the non-negativity of $\Delta(2, c_1, c_2)$ suffices for the existence of a holomorphic rank-2 vector bundle E on X with $c_1(E) = c_1$ and $c_2(E) = c_2$. This problem was addressed for all non-algebraic surfaces in [3]. For primary Kodaira surfaces, a complete solution was given in [1].

Related to this problem is the notion of filtrability: a rank-2 torsion-free sheaf on X is called *filtrable* if it has a rank-one coherent subsheaf (in higher ranks one would ask for a filtration with terms in every rank). On a non-algebraic surface X, filtrable rank-2 vector bundles with Chern classes c_1 and c_2 exist if and only if

$$\Delta(2, c_1, c_2) \geq m(2, c_1),$$

where

$$m(2, c_1) := -\frac{1}{2}\max\left\{\left(\frac{c_1}{2} - \mu\right)^2 \ : \ \mu \in \mathrm{NS}(X)\right\}$$