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Global existence of solutions of the liquid crystal flow for the Oseen–Frank model in $\mathbb{R}^2$

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Abstract

In the first part of this paper, we establish the global existence of solutions of the liquid crystal (gradient) flow for the well-known Oseen–Frank model. The liquid crystal flow is a prototype of equations from the Ericksen–Leslie system in the hydrodynamic theory and generalizes the heat flow for harmonic maps into the 2-sphere. The Ericksen–Leslie system is a system of the Navier–Stokes equations coupled with the liquid crystal flow. In the second part of this paper, we also prove the global existence of solutions of the Ericksen–Leslie system for a general Oseen–Frank model in $\mathbb{R}^2$.

MSC: 35K50; 35Q30

Keywords: Liquid crystals flow; Navier–Stokes equations

1. Introduction

A *liquid crystal* is a state of matter intermediate between a crystalline solid and a normal isotropic liquid. Research into liquid crystals is an area of a very successful synergy between mathematics and physics. There are a lot of analytical and computational issues, which arise in the attempt to study static equilibrium configurations. Numerical and experimental analysis has

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shown that equilibrium configurations are expected to have point and line singularities [21]. Mathematically, Hardt et al. in their fundamental papers [15,16] proved the existence of an energy minimizer u of the liquid crystal functional and showed that a minimizer u is smooth away from a closed set Σ of Ω. Moreover, Σ has Hausdorff dimension strictly less than one. In [1], Almgren and Lieb did some related analysis indicating that the phenomenon is of wider interest. In physical theory, an equilibrium configuration corresponds to a critical point, not necessarily an energy minimizer, of the liquid crystal energy. Critical points are much harder to understand mathematically than minima. From the above result of Hardt et al., minimizers cannot have line singularities. Following the work of Bethuel–Brezis–Coron on harmonic maps in [4], Giaquinta et al. [12] found a relaxed energy for the liquid crystal systems, whose minimizers are also equilibrium configurations. On the other hand, Giaquinta et al. [11] also proved that minimizers of the relaxed energy for harmonic maps are smooth away from a 1-dimensional singular set. Further developments on the regularity results on harmonic maps were surveyed in [13]. There is an interesting open problem to prove that minimizers of the relaxed liquid crystal energy have line singularities. The first author in [17] proved the partial regularity of minimizers of the modified relaxed energy of the liquid crystal energy. However, the partial regularity of minimizers of the relaxed energy for liquid crystals is still mysterious. In some related studies of liquid crystals, Bauman et al. [3] studied the Landau–de Gennes free energy used to describe the transition between chiral nematic and the smectic liquid crystal phase, Lin and Pan [29] used the Landau–de Gennes models to investigate the magnetic field induced instabilities in liquid crystals, and the existence of infinitely many liquid crystal equilibrium configurations prescribing the same boundary was obtained in [18].

A general description of the static theory of liquid crystals is given by Ericksen in [9]. A liquid crystal is composed of rod like molecules which display orientational order, unlike a liquid, but lacking the lattice structure of a solid. The kinematic variable in the nematic and the cholesteric phase may be taken to the optic axis, which is a unit vector field u in a region $\Omega \subset \mathbb{R}^3$ occupied by the materials. The liquid crystal energy for a configuration $u \in H^1(\Omega; S^2)$ is given by

$$E(u;\Omega) = \int_\Omega W(u,\nabla u)\,dx, \tag{1.1}$$

where the Oseen–Frank density $W(u,\nabla u)$, depending on positive material constants k_1, k_2, k_3 and k_4, is given by

$$W(u,\nabla u) = k_1(\operatorname{div} u)^2 + k_2(u\cdot \operatorname{curl} u)^2 + k_3|u\times \operatorname{curl} u|^2 + k_4[\operatorname{tr}(\nabla u)^2 - (\operatorname{div} u)^2].$$

Without loss of generality, as in [15] or [13], we rewrite the density

$$W(u,\nabla u) = a|\nabla u|^2 + V(u,\nabla u), \quad a = \min\{k_1,k_2,k_3\} > 0, \tag{1.2}$$

where

$$V(u,\nabla u) = (k_1 - a)(\operatorname{div} u)^2 + (k_2 - a)(u\cdot \operatorname{curl} u)^2 + (k_3 - a)|u\times \operatorname{curl} u|^2.$$

A static equilibrium configuration corresponds to an extremal (critical point) of the energy functional E in $H^1(\Omega, S^2)$. The Euler–Lagrange system for the general Oseen–Frank functional (1.1) (see details in the Appendix of Section 5) is

$$\begin{aligned}\nabla_\alpha\Big[W_{p^i_\alpha}(u,\nabla u) - u^l u^i V_{p^l_\alpha}(u,\nabla u)\Big] - W_{u^i}(u,\nabla u) + W_{u^l}(u,\nabla u)u^l u^i \\ + W_{p^l_\alpha}(u,\nabla u)\nabla_\alpha u^l u^i + V_{p^l_\alpha}(u,\nabla u)u^l\nabla_\alpha u^i = 0 \quad \text{in } \Omega\end{aligned} \tag{1.3}$$