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Support properties for integral operators in hyperfunctions

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Abstract

The main result of this paper is that the integral operators between spaces of compactly supported hyperfunctions must have properly supported kernels. We also discuss the uniqueness and the regularity of the integral operators in hyperfunction theory.

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1. Statement of the main result

Consider open sets $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, and a hyperfunction $\mathcal{K}$ defined on $V \times U$ satisfying the condition

$$\{(x, y, 0, \eta) \in V \times U \times \mathbb{R}^n \times \mathbb{R}^m;\, \eta \neq 0\} \cap \mathrm{WF}_A \mathcal{K} = \emptyset, \tag{1.1}$$

where $\mathrm{WF}_A \mathcal{K}$ denotes the analytic wave front set of $\mathcal{K}$. We can then associate with $\mathcal{K}$ a linear operator $T : \mathcal{A}'(U) \to \mathcal{B}(V)$, by

$$(Tu)(x) = \int_U \mathcal{K}(x, y)u(y)\, dy, \quad \text{for } u \in \mathcal{A}'(U). \tag{1.2}$$

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Here $\mathcal{A}'(U)$ denotes the space of real-analytic functionals on $U, \mathcal{B}(V)$ the space of hyperfunctions on V and the meaning of the integral in (1.2) is the one given by microlocal analysis to such expressions. (Cf. [18,9]. We shall have to come back to this in Section 4.) Note that we shall identify $\mathcal{A}'(U)$ with the space $\mathcal{B}_c(U)$ of hyperfunctions on U with compact support. In this setting, T is said to be the integral operator associated with $\mathcal{K}$, and $\mathcal{K}$ is said to be the kernel of T.

Our main result is the following:

Theorem 1.1. *Consider $\mathcal{K} \in \mathcal{B}(V \times U)$ satisfying (1.1) and let $T : \mathcal{A}'(U) \to \mathcal{B}(V)$ be the associated integral operator. The following conditions are equivalent:*

(i) $T(\mathcal{A}'(U)) \subset \mathcal{A}'(V)$.
(ii) *For every compact set $K \subset U$ there is a compact set $L \subset V$ such that* $\operatorname{supp} u \subset K$ *implies* $\operatorname{supp} Tu \subset L$.
(iii) *The map $p_2|_{\operatorname{supp}\mathcal{K}} : \operatorname{supp}\mathcal{K} \to U$ is proper, where p_2 denotes the second projection $V \times U \to U$.*
(iv) *T is a composition of a continuous linear map $\mathcal{A}'(U) \to \mathcal{A}'(V)$ and the inclusion map $\mathcal{A}'(V) \to \mathcal{B}(V)$.*

A kernel $\mathcal{K}$ satisfying (iii) in the theorem above is called a properly supported kernel in this paper.

Theorem 1.1, or more precisely speaking, the implications (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) have been announced in [13] (see Theorem 3.3(1)).

Note that the assumption in (i) is that for every fixed $u \in \mathcal{A}'(U)$ there is a compact set $L \subset V$ with $\operatorname{supp} Tu \subset L$. The new information in (ii) is then just that this compact set L essentially only depends on the support of u and not on u itself. It is nevertheless the implication (i) $\Rightarrow$ (ii) which seems most interesting to us. In fact the main technical difficulty in the proof of this implication is that it is not immediate how to use in a quantitative way (when u is varying) the information that Tu has compact support. At the origin of this is (by the very definition of hyperfunctions) the fact that Tu vanishes in a neighborhood of some point x^0 gives only a cohomological information about the holomorphic representation functions of Tu near x^0.

By contrast, the implications (ii) $\Rightarrow$ (iii), respectively (iii) $\Rightarrow$ (iv), are relatively easy consequences of known results and the fact that (iv) implies (i) is of course trivial. Note also that the implication (iii) $\Rightarrow$ (ii) is a direct corollary of the definition of integration along fibers for hyperfunctions. Moreover, we should mention that using functional analysis it is quite easy to prove directly that (iv) $\Rightarrow$ (ii). (See Proposition 3.6.)

We also mention the following results concerning the uniqueness and the regularity of the kernels.

Theorem 1.2. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying (1.1) and denote by T the associated operator defined in (1.2). If $Tu = 0$ for every $u \in \mathcal{A}'(U)$, then $\mathcal{K}$ must vanish on $V \times U$.*

Theorem 1.3. *Let $\mathcal{K} \in \mathcal{B}(V \times U)$ be a hyperfunction satisfying (1.1). Assume that the operator $T : \mathcal{A}'(U) \to \mathcal{B}(V)$ defined in (1.2) actually maps $\mathcal{A}'(U)$ into $\mathcal{A}(V)$. Then $\mathcal{K}$ is real-analytic on $V \times U$.*