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Characterizing the strongly jump-traceable sets via randomness

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Abstract

We show that if a set A is computable from every superlow 1-random set, then A is strongly jump-traceable. Together with a result of Greenberg and Nies [Noam Greenberg, André Nies, Benign cost functions and lowness properties, J. Symbolic Logic 76 (1) (2011) 289–312], this theorem shows that the computably enumerable (c.e.) strongly jump-traceable sets are exactly the c.e. sets computable from every superlow 1-random set.

We also prove the analogous result for superhighness: a c.e. set is strongly jump-traceable if and only if it is computable from every superhigh 1-random set.

Finally, we show that for each cost function c with the limit condition there is a 1-random Δ^0_2 set Y such that every c.e. set $A \leqslant_{\mathrm{T}} Y$ obeys c. To do so, we connect cost function strength and the strength of randomness notions. Together with a theorem of Greenberg and Nies (ibd.), this result gives a full correspondence between obedience of cost functions and being computable from Δ^0_2 1-random sets.

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1. Background and motivation

There are two aspects to the information content of sets of natural numbers. In terms of computational complexity, a set of numbers is considered to code a lot of information if it is useful as an oracle for relative computation. In terms of effective randomness, difficulty to detect patterns in the set marks it as complicated, or random. The interaction between these two aspects of complexity is the focus of much current research in computability theory.

Although earlier research naturally gravitated toward the complex, recent findings have shown rich structure in the region of the simple. Properties of sets that indicate being uncomplicated are called *lowness properties*. They have proved to be essential in the understanding of random sets, and of the connections between computability and randomness along the entire spectrum of complexity.

A lowness property that is central to this study is that of K-triviality. A series of results by Downey, Hirschfeldt, Nies, and Stephan (see [18,31]) developed penetrating techniques for the study of several classes of low sets. These results have established the coincidence of several such notions, three of the important ones being: K-triviality (being far from random), lowness for randomness (not being able to detect new patterns in random sets), and being computable from a relatively random oracle. (Here, "random" means Martin-Löf random, or 1-random, as defined in Section 2.3.) This coincidence established the robustness of this class. Further results have demonstrated its usefulness and importance to the field; see, for example, [11].

The diverse characterizations of the K-trivial sets, and the techniques used to study them, have led to three paradigms for understanding lowness of a set A of natural numbers, introduced by Nies [34,35].

1. *Being weak as an oracle.* This paradigm means that A is not very useful as an oracle for Turing machines. This is the oldest way of thinking about lowness. For instance, A is of hyperimmune free degree if it does not compute fast growing functions: each function computed by A is dominated by a computable function. Some formal instances of the paradigm are expressed through A', the halting set relative to A. For instance, the traditional notion, simply called "low", states that A' is as simple as possible in the Turing degrees. The newer notion of superlowness states that A' is as simple as possible in the truth-table degrees.

2. *Being computed by many oracles.* Traditionally, there were no interesting answers to the question "how many sets compute A?"; the answer is always "uncountably many"—indeed continuum many—but unless A is computable (in which case every set computes A), the collection of sets computing A has measure 0. Recently, more detailed answers have proved to be insightful, in particular in conjunction with answers to the question "what *kinds* of sets compute A?" For example, as noted above, A is K-trivial if and only if A is computed by some set that is 1-random relative to A, in which case the class of oracles computing A is large in an effective sense relative to A.

3. *Being inert.* Shoenfield's limit lemma states that a set A is computable from the halting set $\emptyset'$ if and only if it has a computable approximation. (We let Δ^0_2 denote the collection of such sets.) The inertness paradigm says that a Δ^0_2 set A is close to computable if it is computably approximable with a small number of changes. For formal instances of the inertness paradigm, we use so-called *cost functions*. They measure the total number of changes of a Δ^0_2 set, and especially that of a computably enumerable set. Most examples of cost functions are based on randomness-related concepts. (Precise definitions of all of these concepts will be given below. For more background on these paradigms see [36].)