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The Hall algebra approach to Drinfeld's presentation of quantum loop algebras

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Abstract

The quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ was defined as a generalization of the Drinfeld's new realization of the quantum affine algebra to the loop algebra of any Kac–Moody algebra $\mathfrak{g}$. It has been shown by Schiffmann that the Hall algebra of the category of coherent sheaves on a weighted projective line is closely related to the quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$, for some $\mathfrak{g}$ with a star-shaped Dynkin diagram. In this paper we study Drinfeld's presentation of $U_v(\mathcal{L}\mathfrak{g})$ in the double Hall algebra setting, based on Schiffmann's work. We explicitly find out a collection of generators of the double composition algebra $\mathbf{DC}(\mathrm{Coh}(\mathbb{X}))$ and verify that they satisfy all the Drinfeld relations.
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1. Introduction

1.1

Let $\mathfrak{g}$ be a Kac–Moody algebra, $U(\mathfrak{g})$ be its universal enveloping algebra. The Drinfeld–Jimbo quantum group $U_v(\mathfrak{g})$ is defined by a collection of generators and relations (see 3.2), which is a

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certain deformation of the Chevalley generators and Serre relations for $U(\mathfrak{g})$. When $\mathfrak{g}$ is affine, it is well-known that $\mathfrak{g}$ can be constructed as (a central extension of) the loop algebra $\mathcal{L}\mathfrak{g}_0$ of some simple Lie algebra $\mathfrak{g}_0$. In this case Drinfeld gave another set of generators and relations of $U_v(\mathfrak{g})$ known as *Drinfeld's new realization* of quantum affine algebras. This new presentation can be treated as a certain deformation of the loop algebra presentation of $\mathfrak{g}$. The isomorphism of the two presentations of $U_v(\mathfrak{g})$ was proved by Beck [2] (also see [14]). One can define the quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ for any Kac–Moody algebra $\mathfrak{g}$ as a generalization of Drinfeld's presentation for quantum affine algebras (see 3.4).

1.2

The Ringel–Hall algebra approach to quantum groups has been developed since the 1990s, which shows a deep relationship between Lie theory and finite dimensional hereditary algebras. More precisely, let Q be the quiver whose underlying graph is the Dynkin diagram of the Kac–Moody algebra $\mathfrak{g}$. Consider the category of finite dimensional representations of Q over a finite field $k = \mathbb{F}_q$, denoted by mod(kQ). Due to Ringel and Green [21,11], the composition subalgebra of the Hall algebra $\mathbf{H}$ (mod kQ) is isomorphic to the positive part of the quantum group $U_v^+(\mathfrak{g})$ where v specializes to $\sqrt{q}$. This result was generalized to the whole quantum group by using the technique of Drinfeld double for Hopf algebras [25] (see 2.5).

Thus it is quite natural to consider the following problem.

Problem 1.1. How to understand Drinfeld's presentation of quantum affine algebras (and more generally, quantum loop algebras) in the Hall algebra setting?

One possible way to solve the problem for quantum affine algebras is to explain Beck's isomorphism in the language of Hall algebras. For type $\tilde{A}$ Hubery has given the answer for the positive part $U_v^+(\widehat{\mathfrak{sl}}_n)$ using nilpotent representations of cyclic quivers [13]. But it seems not easy to generalize his method to other types. We should also mention that McGerty [19] has given the Drinfeld generators for the positive part $U_v^+(\widehat{\mathfrak{sl}}_2)$ using representations of the Kronecker quiver.

1.3

On the other hand, in his remarkable paper [15] Kapranov observed that there are connections between the Hall algebra of the category of coherent sheaves on a smooth projective curve X and Drinfeld's new realization of the quantum affine algebra. And when X is the projective line, he constructed an isomorphism between a subalgebra of the Hall algebra and another positive part (compared with the standard one, see 3.2) of $U_q(\widehat{\mathfrak{sl}}_2)$ (also see [1]).

This result was generalized by Schiffmann [23] using Hall algebras of the categories of coherent sheaves on weighted projective lines (introduced in [10]) as follows.

When the Dynkin diagram Γ of $\mathfrak{g}$ is a star-shaped graph, he defined a certain positive part $U_v(\hat{\mathfrak{n}})$ of the quantum loop algebra $U_v(\mathcal{L}\mathfrak{g})$ (note that there is no standard positive part of $U_v(\mathcal{L}\mathfrak{g})$ since in general the loop algebra $\mathcal{L}\mathfrak{g}$ is not a Kac–Moody algebra). And he established an epimorphism from $U_v(\hat{\mathfrak{n}})$ to a subalgebra $\mathbf{C}(\mathrm{Coh}\,\mathbb{X})$ of the Hall algebra $\mathbf{H}(\mathrm{Coh}\,\mathbb{X})$, where $\mathbb{X}$ is the weighted projective line associated to Γ. Moreover, when $\mathbb{X}$ is of parabolic or elliptic type (the corresponding $\mathfrak{g}$ is of finite or affine type), the epimorphism is an isomorphism (see Theorem 5.4). This means that the Hall algebra for weighted projective lines is the right framework to consider Problem 1.1 for general quantum loop algebras.

However, the problem was not completely solved in Schiffmann's work. Namely not all Drinfeld's generators and relations were explicitly found out in the corresponding Hall algebra.

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