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Advances in Mathematics 231 (2012) 2626–2665

ADVANCES IN Mathematics

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The extremal index, hitting time statistics and periodicity

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Received 12 August 2011; accepted 12 July 2012
Available online 7 September 2012

Communicated by D. Stroock

Abstract

The extremal index appears as a parameter in Extreme Value Laws for stochastic processes, characterising the clustering of extreme events. We apply this idea in a dynamical systems context to analyse the possible Extreme Value Laws for the stochastic process generated by observations taken along dynamical orbits with respect to various measures. We derive new, easily checkable, conditions which identify Extreme Value Laws with particular extremal indices. In the dynamical context we prove that the extremal index is associated with periodic behaviour. The analogy of these laws in the context of hitting time statistics, as studied in the authors' previous works on this topic, is explained and exploited extensively allowing us to prove, for the first time, the existence of hitting time statistics for balls around periodic points. Moreover, for very well behaved systems (uniformly expanding) we completely characterise the extremal behaviour by proving that either we have an extremal index less than 1 at periodic points or equal to 1 at any other point. This theory then also applies directly to general stochastic processes, adding both useful tools to identify the extremal index and giving deeper insight into the periodic behaviour it suggests.

MSC: 60G70; 60G10; 37A50; 37B20; 37C25

Keywords: Extremal index; Extreme value theory; Return time statistics; Stationary stochastic processes; Periodicity

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.07.029

1. Introduction

The study of extreme or rare events is of great importance in a wide variety of fields and is often tied in with risk assessment. This explains why Extreme Value Laws (EVLs) and the estimation of the tail distribution of the maximum of a large number of observations has drawn much attention and become a highly developed subject.

In many practical situations, such as in the analysis of financial markets or climate phenomena, time series can be modelled by a dynamical system which describes its time evolution. The recurrence effect introduced by Poincaré, which is present in chaotic systems, is the starting point for a deeper analysis of the limit distribution of the elapsed time until the occurrence of a rare event, which is usually referred to as Hitting Time Statistics (HTS) and Return Time Statistics (RTS).

In [24], we established the connection between the existence of EVL and HTS/RTS for stochastic processes arising from discrete time chaotic dynamical systems. This general link allowed us to obtain results of EVL using tools from HTS/RTS and the other way around (this was applied in cases where the extremal index was 1, which is the most classical setting).

The Extremal Index (EI) $\theta \in [0, 1]$ is a measure of clustering of extreme events, the lower the index, the higher the degree of clustering. In this paper, we give general conditions to prove the existence of an extremal index $0 < \theta < 1$, which can be applied to any stationary stochastic process. Although our results apply to general stationary stochastic processes, we will be particularly interested in the case where the stochastic process arises from a discrete time dynamical system. This setup will provide not only a huge diversity of examples, but also a motivation for the conditions we propose, as well as a better understanding of their implications. Namely, motivated by the study of stochastic processes arising from chaotic dynamical systems, we associate the extremal index to the occurrence of periodic phenomena. We will illustrate these results by applying them to time series provided by deterministic dynamical systems as well as to cases where the extremal index is already well understood: an Autoregressive (AR) process introduced by Chernick and two Maximum Moving Average (MMA) processes.

Because our conditions on the time series data which guarantee an EVL with a given EI are so general, in the dynamical systems context we are able to prove strong results on EVLs around periodic points. For example, this allows us to consider non-uniformly hyperbolic dynamical systems. Moreover, coupling these weak conditions with the connection of EVLs to HTS/RTS enables us to consider hits/returns to balls, rather than cylinders. To our knowledge this is the first result of HTS/RTS different from the standard exponential which applies to balls. We do this first for so-called ‘Rychlik systems’ which are a very general form of uniformly expanding interval map. As explained in Remark 5, these results can easily be extended to some higher dimensional version of these Rychlik systems. We also give an example of non-uniformly hyperbolic dynamical system: the full quadratic map (also known as the quadratic Chebyshev polynomial), where invariant measures are absolutely continuous w.r.t. Lebesgue or are, more generally, equilibrium states w.r.t. certain potentials. In future work we will apply these ideas to even more badly behaved non-uniformly hyperbolic systems.

One of the striking results here is that, at least for well-behaved systems, an extremal index different from 1 can *only* occur at periodic points. We prove this for the full shift equipped with the Bernoulli measure. (We believe that this last result holds in greater generality, but do not prove that here). Hence, this result raises the following.

Question. *Is it possible to prove the existence of an EI in* $(0, 1)$ *without some sort of periodicity?*