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On a conjecture of Anosov

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Abstract

In this paper, we prove that for every bumpy Finsler n-sphere (S^n, F) with reversibility λ and flag curvature K satisfying $(\frac{\lambda}{\lambda+1})^2 < K \le 1$, there exist $2[\frac{n+1}{2}]$ prime closed geodesics. This gives a confirmed answer to a conjecture of Anosov [1] in 1974 for a generic case.

MSC: 53C22; 53C60; 58E10

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1. Introduction and main results

This paper is devoted to a study on closed geodesics on Finsler spheres. Let us recall first the definition of Finsler metrics.

Definition 1.1 (*Cf. [5,29]*)**.** Let M be a finite dimensional smooth manifold. A function $F : TM \to [0, +\infty)$ is a Finsler metric if it satisfies

(F1) F is C^∞ on $TM \setminus \{0\}$,
(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_xM$, $x \in M$, and $\lambda > 0$,
(F3) for every $y \in T_xM \setminus \{0\}$, the quadratic form

$$g_{x,y}(u, v) \equiv \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \quad \forall u, v \in T_xM,$$

is positive definite.

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In this case, (M, F) is called a Finsler manifold. F is reversible if $F(x, -y) = F(x, y)$ holds for all $y \in T_xM$ and $x \in M$. F is Riemannian if $F(x, y)^2 = \frac{1}{2}G(x)y \cdot y$ for some symmetric positive definite matrix function $G(x) \in GL(T_xM)$ depending on $x \in M$ smoothly.

A closed curve in a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [29]). As usual, on any Finsler n-sphere $S^n = (S^n, F)$, a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to S^n$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1-t)$ for $t \in \mathbf{R}$. Note that on a non-symmetric Finsler manifold, the inverse curve of a closed geodesic is not a closed geodesic in general. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t+\theta)$ for all $t \in \mathbf{R}$. We shall omit the word *distinct* when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) n-sphere, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., their image sets in S^n are distinct.

For a closed geodesic c on (S^n, F), denote by P_c the linearized Poincaré map of c. Then $P_c \in \mathrm{Sp}(2n-2)$ is symplectic. For any $M \in \mathrm{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$ in the complex plane $\mathbf{C}$. Since M is symplectic, $e(M)$ is even and $0 \le e(M) \le 2k$. A closed geodesic c is called *elliptic* if $e(P_c) = 2(n-1)$, i.e., all the eigenvalues of P_c locate on $\mathbf{U}$; *hyperbolic* if $e(P_c) = 0$, i.e., all the eigenvalues of P_c locate away from $\mathbf{U}$; *non-degenerate* if 1 is not an eigenvalue of P_c. A Finsler sphere (S^n, F) is called *bumpy* if all the closed geodesics on it are non-degenerate.

Following Rademacher in [25], the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold (M, F) is defied to be

$$\lambda := \max\{F(-X) \mid X \in TM,\ F(X) = 1\} \ge 1.$$

It was quite surprising when Katok [12] in 1973 found some non-reversible Finsler metrics on CROSS with only finitely many prime closed geodesics and all closed geodesics are non-degenerate and elliptic. The smallest number of closed geodesics on S^n that one obtains in these examples is $2[\frac{n+1}{2}]$ (cf. [32]). Then Anosov in I.C.M. of 1974 conjectured that the lower bound of the number of closed geodesics on any Finsler sphere (S^n, F) should be $2[\frac{n+1}{2}]$, i.e., the number of closed geodesics in Katok's example.

We can show that under some conditions this conjecture is true, i.e., the following main result of the paper.

Theorem 1.2. *For every bumpy Finsler n-sphere (S^n, F) with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \le 1$, there exist $2[\frac{n+1}{2}]$ prime closed geodesics.*

We can obtain a stability result.

Theorem 1.3. *For every bumpy Finsler n-sphere (S^n, F) with reversibility λ and flag curvature K satisfying $\left(\frac{3\lambda}{2(\lambda+1)}\right)^2 < K \le 1$, there exist two elliptic prime closed geodesics provided the number of prime closed geodesics on (S^n, F) is finite.*

Remark 1.4. Note that Katok metrics on S^n has constant flag curvature 1 (cf. p. 764 in [27]). In [3], Ballmann et al. proved that for a Riemannian metric on S^n with sectional curvature