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Differential operators on quantized flag manifolds at roots of unity

Toshiyuki Tanisaki

Department of Mathematics, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

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Abstract

The quantized flag manifold, which is a q-analogue of the ordinary flag manifold, is realized as a non-commutative scheme, and we can define the category of D-modules on it using the framework of non-commutative algebraic geometry; however, when the parameter q is a root of unity, Lusztig's Frobenius morphism allows us to handle D-modules on the quantized flag manifold through modules over a certain sheaf of rings on the ordinary flag manifold. In this paper, we will show that this sheaf of rings on the ordinary flag manifold is an Azumaya algebra over its center. We also show that its restriction to certain subsets are split Azumaya algebras. These are analogues of some results of Bezrukavnikov–Mirković–Rumynin on D-modules on flag manifolds in positive characteristics.

MSC: 20G05; 17B37

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0. Introduction

0.1

In [20] Lunts and Rosenberg constructed the quantized flag manifold for a quantized enveloping algebra as a non-commutative projective scheme. They also defined a category of

E-mail address: tanisaki@sci.osaka-cu.ac.jp.

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D-modules on it, and conjectured a Beilinson–Bernstein type equivalence of categories. In [29] we proposed a modification of the definition of the ring of differential operators on the quantized flag manifold, and established a Beilinson–Bernstein type equivalence for the modified ring of differential operators (see also Backelin–Kremnizer [3]).

The above mentioned results are for a quantized enveloping algebra when the parameter q is transcendental. The aim of this paper is to investigate the ring of differential operators on the quantized flag manifold when the parameter is a root of unity. It is a general phenomenon that quantized objects at roots of unity resemble ordinary objects in positive characteristics. Hence it is natural to pursue analogue of the theory of D-modules on flag manifolds in positive characteristics due to Bezrukavnikov et al. [5]. In [5] an analogue of the Beilinson–Bernstein equivalence was established on the level of derived categories. Moreover, it was also shown there that the ring of differential operators satisfies certain Azumaya properties. In this paper we will be concerned with the Azumaya properties in the quantized situation.

0.2

Let G be a connected simply-connected simple algebraic group over $\mathbb{C}$, and let $\mathfrak{g}$ be its Lie algebra. We fix Borel subgroups B^+ and B^- of G such that $H = B^+ \cap B^-$ is a maximal torus of G. We denote by $N^{\pm}$ the unipotent radical of $B^{\pm}$. We denote by Q and Λ the root lattice and the weight lattice respectively. We also denote by Λ^+ the set of dominant weights. Set $\mathbb{F} = \mathbb{Q}(q^{1/|\Lambda/Q|})$, where $q^{1/|\Lambda/Q|}$ is an indeterminate. We denote by $U_{\mathbb{F}}$ the quantized enveloping algebra of $\mathfrak{g}$ over $\mathbb{F}$. It is a Hopf algebra over $\mathbb{F}$, and is generated as an $\mathbb{F}$-algebra by the elements k_λ, e_i, f_i ($\lambda \in \Lambda, i \in I$), where I is the index set for simple roots for $\mathfrak{g}$. We can define a q-analogue $C_{\mathbb{F}}$ of the coordinate algebra of G as a Hopf algebra dual of $U_{\mathbb{F}}$. More precisely, we define $C_{\mathbb{F}}$ to be the subspace of $\mathrm{Hom}_{\mathbb{F}}(U_{\mathbb{F}}, \mathbb{F})$ spanned by the matrix coefficients of type 1 representations of $U_{\mathbb{F}}$. Then we have a $U_{\mathbb{F}}$-bimodule structure of $C_{\mathbb{F}}$ given by

$$\langle u_1 \cdot \varphi \cdot u_2, u\rangle = \langle \varphi, u_2 u u_1\rangle \quad (u, u_1, u_2 \in U_{\mathbb{F}}, \varphi \in C_{\mathbb{F}}).$$

Set

$$A_{\mathbb{F}} = \bigoplus_{\lambda \in \Lambda^+} A_{\mathbb{F}}(\lambda) \subset C_{\mathbb{F}}$$

with

$$A_{\mathbb{F}}(\lambda) = \{\varphi \in C_{\mathbb{F}} \mid \varphi \cdot v = \chi_\lambda(v)\varphi \ (v \in U_{\mathbb{F}}^{\leqq 0})\},$$

where $U_{\mathbb{F}}^{\leqq 0}$ is the subalgebra of $U_{\mathbb{F}}$ generated by k_λ, f_i ($\lambda \in \Lambda, i \in I$) and χ_λ is the character of $U_{\mathbb{F}}^{\leqq 0}$ corresponding to λ. Note that $A_{\mathbb{F}}$ is a non-commutative Λ-graded $\mathbb{F}$-algebra. The quantized flag manifold $\mathcal{B}_q$ is defined as a non-commutative projective scheme by

$$\mathcal{B}_q = \mathrm{Proj}_\Lambda(A_{\mathbb{F}}).$$

This actually means that we are given an abelian category $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_q})$ of "quasi-coherent sheaves on $\mathcal{B}_q$" defined by

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_q}) = \mathrm{Mod}_\Lambda(A_{\mathbb{F}})/\mathrm{Tor}_{\Lambda^+}(A_{\mathbb{F}}),$$

where $\mathrm{Mod}_\Lambda(A_{\mathbb{F}})$ is the category of Λ-graded left $A_{\mathbb{F}}$-modules, and $\mathrm{Tor}_{\Lambda^+}(A_{\mathbb{F}})$ denotes its full subcategory consisting of $M \in \mathrm{Mod}_\Lambda(A_{\mathbb{F}})$ such that for each $m \in M$ there exists some $\lambda \in \Lambda^+$ such that $A_{\mathbb{F}}(\lambda + \mu)m = 0$ for any $\mu \in \Lambda^+$.