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Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds

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Abstract

This article mainly concerns with the non-existence, existence, and multiplicity results for positive solutions to the Einstein-scalar field Lichnerowicz equation on closed manifolds with a negative conformal-scalar field invariant. This equation arises from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity. Our analysis introduces variational techniques to the analysis of the Hamiltonian constraint equation, especially those cases when the prescribed scalar curvature-scalar field function may change sign. To our knowledge, such a problem remains open.

MSC: 35J60; 53C21; 53C80; 58E05; 58J45; 83C05

Keywords: Einstein-scalar field equation; Lichnerowicz equation; Picone type identity; Critical exponent; Negative exponent; Mountain pass theorem; Palais–Smale condition; Sign-changing nonlinearity

Contents

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1. Introduction

Along with the rapid development in general relativity, physicists pose many challenging problems to mathematicians, for example, the initial value problems, the well-posedness problems, the global stability problems, etc. Among these problems, the initial value problem turns out to be the most interesting problem from the mathematical point of view. When solving the initial value problems, one needs to solve the so-called constraint equations which can be formulated via the following system of equations defined on a Riemannian manifold $(M, \overline{g})$ without the boundary of dimension $n \geqslant 3$,

$$\begin{aligned} &\mathrm{Scal}_{\overline{g}} - |\overline{K}|^2_{\overline{g}} + (\mathrm{trace}_{\overline{g}}\overline{K})^2 - 2\rho = 0, \\ &\nabla_{\overline{g}} \cdot \overline{K} - \nabla_{\overline{g}}\mathrm{trace}_{\overline{g}}\overline{K} - J = 0, \end{aligned} \tag{1.1}$$

where all quantities of (1.1) involving a metric are computed with respect to $\overline{g}$, an induced metric of $\mathbf{g}$ when embedded in a spacetime $(V, \mathbf{g})$, $\overline{K}$ the second fundamental form, $\mathrm{Scal}_{\overline{g}}$ the scalar curvature of $\overline{g}$, ρ a scalar, J a vector field on M, and T a tensor of the sources; see [6,7,9].

Since the constraint equations form an under-determined system, they are in general hard to solve. However, it was remarked in [6] that the conformal method can be effectively applied in the constant mean curvature setting, that is to look for the metric $\overline{g}$ of the form $u^{\frac{4}{n-2}} g$ where g is fixed. To be precise, when the conformal method is applied in this setting, the constraint equations (1.1) are easily transformed to the so-called Hamiltonian and momentum constraints. In the literature, the momentum constraint is a second-order semilinear elliptic equation that can be easily solved if we are in the constant mean curvature setting. The most difficult part is to solve the Hamiltonian constraint which can be formulated by a simple partial differential equation,

$$\Delta_g u + hu = f u^{2^\star - 1} + \frac{a}{u^{2^\star + 1}}, \quad u > 0, \tag{1.2}$$