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# A Faber–Krahn inequality for solutions of Schrödinger's equation

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**Abstract**

We consider nontrivial solutions of $-\Delta u(x) = V(x)u(x)$, where $u \equiv 0$ on the boundary of a bounded open region $D \subset \mathbf{R}^n$, and $V(x) \in L^\infty(D)$. We prove a sharp relationship between $\|V\|_\infty$ and the measure of $D$, which generalizes the well-known Faber–Krahn theorem. We also prove some geometric properties of the zero sets of the solution of the Schrödinger equation $-\Delta u(x) = V(x)u(x)$.
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## 1. Introduction

We study nontrivial solutions of the Schrödinger equation

$$-\Delta u(x) = V(x)u(x), \tag{1.1}$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial_j^2}$, which vanish on the boundary of a bounded open region $D \subset \mathbf{R}^n$, $n \geq 1$. We say that $u$ is nontrivial if it does not vanish identically in $D$.

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We establish a sharp relationship between the potential $V$ and the measure of $D$. Let $B(0,1)$ denote the unit ball in $\mathbf{R}^n$, and set $\omega_n = |B(0,1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$. Let $j = j_{\frac{n}{2}-1}$ be the first zero of the Bessel function $J_{\frac{n}{2}-1}(x)$. Our main result is:

**Theorem 1.1.** *Suppose that $u \in C(\overline{D})$ is a nontrivial solution of (1.1) in the distribution sense. Suppose that $u \equiv 0$ on $\partial D$, and $V \in L^\infty(D)$. Then*

$$|D| \cdot \|V\|_\infty^{\frac{n}{2}} \geq j^n \omega_n. \tag{1.2}$$

We show below that dilations and constant multiples of

$$u_*(x) = |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|), \tag{1.3}$$

where $J_a(r)$ is the Bessel function of the first kind, give equality in (1.2), so the constant $C = j^n \omega_n$ in the theorem is sharp. In the proof of this theorem, and of related ones in Section 2, we can assume without loss of generality that $u > 0$ on $D$; if $u$ changes sign on $D$, we can apply the theorem on the subset where $u > 0$ instead. Note that the formula $|D| \cdot \|V\|_\infty^{\frac{n}{2}}$ is dilation-invariant, so we may also assume that $\|V\|_\infty = 1$. In [4], the authors proved (1.2), but with a smaller constant $c$. When $n = 2$, for example, we obtained $c = 4\pi$; the constant in (1.2) is $C = \pi j^2$ with $j \sim 2.4048$.

When $V(x) \equiv \lambda$, a constant, $u$ is an eigenfunction for the Dirichlet problem:

$$\begin{cases} -\Delta u(x) = \lambda u(x) & x \in D \\ u \equiv 0 & x \in \partial D. \end{cases} \tag{1.4}$$

The well-known Faber–Krahn inequality (see e.g. [2]) states that, for any bounded domain $D$ of fixed volume $|D|$, the smallest possible eigenvalue of the Dirichlet problem (1.4) occurs when $D$ is a ball. That is, if $D^*$ is the ball centered at the origin with $|D| = |D^*|$, and $\lambda_1(D)$ is the first eigenvalue of the Dirichlet problem (1.4), then $\lambda_1(D) \geq \lambda_1(D^*)$. When $D = B(0,1)$, the smallest eigenvalue of (1.4) is $\lambda_1(D) = j^2$, and the eigenfunctions are constant multiples of $u_*(jx)$, where $u_*$ and $j$ are defined as in Theorem 1.1. Thus, our result generalizes the Faber–Krahn result, with the same extremals. Another interesting generalization of the Faber–Krahn inequality appears in [10]; assuming that $\partial D$ is smooth, $|D|$ is fixed, and $v : D \to \mathbf{R}^n$ is bounded, the smallest possible eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta u(x) + v.\nabla u = \lambda u(x) & x \in D \\ u \equiv 0 & x \in \partial D, \end{cases}$$

occurs when $D$ is a ball and $v$ is constant.

Neither our Theorem 1.1 nor the Faber–Krahn inequality applies on unbounded domains in $\mathbf{R}^2$; a counterexample is given in Section 2. Our proof fails in this case mainly because it depends on Green's identity. However, with mild assumptions on $u$ at infinity, both theorems hold when $D$ is unbounded. See Theorem 2.7. One can slightly relax the assumption that $u$ vanishes on the boundary, and then prove (1.2) with a slightly smaller constant; see Proposition 2.4 in Section 2. This is a key idea in the proofs of Theorems 1.1 and 2.7.

One can easily apply Theorem 1.1 to eigenfunctions of the operator $-\Delta - V$. Such a function satisfies $-\Delta u(x) - V(x)u(x) = \lambda u(x)$. By setting $V_2 = V + \lambda$, we are back to (1.1), with a different potential, and Theorem 1.1 implies $|D| \cdot \|V + \lambda\|_\infty^{\frac{n}{2}} \geq j^n \omega_n$.

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