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Advances in Mathematics 231 (2012) 213–242

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Cauchy transform and Poisson's equation

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Received 8 August 2010; accepted 2 May 2012
Available online 31 May 2012

Communicated by C. Kenig

Abstract

Let $u \in W^{1,p} \cap W_0^{1,p}$, $1 \leqslant p \leqslant \infty$ be a solution of the Poisson equation $\Delta u = h$, $h \in L^p$, in the unit disk. We prove $\|\nabla u\|_{L^p} \leqslant a_p \|h\|_{L^p}$ and $\|\partial u\|_{L^p} \leqslant b_p \|h\|_{L^p}$ with sharp constants a_p and b_p, for $p = 1$, $p = 2$, and $p = \infty$. In addition, for $p > 2$, with sharp constants c_p and C_p, we show $\|\partial u\|_{L^\infty} \leqslant c_p \|h\|_{L^p}$ and $\|\nabla u\|_{L^\infty} \leqslant C_p \|h\|_{L^p}$. We also give an extension to smooth Jordan domains.

These problems are equivalent to determining a precise value of the L^p norm of the *Cauchy transform of Dirichlet's problem*.

MSC: primary 35J05; secondary 47G10

Keywords: Möbius transformations; Poisson equation; Newtonian potential; Cauchy transform; Bessel function

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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.05.003

1. Introduction

1.1. Notation

By **U**, we mean the unit disk in the complex plane $\mathbb{C}$ and by **T** its boundary. Throughout the paper Ω denotes a bounded domain in $\mathbb{C}$,

$$dA(z) = dxdy \quad (z = x + iy),$$

the Lebesgue area measure in Ω and

$$d\mu(z) = \frac{1}{\pi} dxdy$$

denotes the normalized area measure in the unit disk **U**.

For $k \geqslant 0$ and $p \geqslant 1$, $W^{k,p}(\Omega)$ is the Banach space of k-times weak differentiable p-integrable functions. The norm in $W^{k,p}(\Omega)$ is defined by

$$\|u\|_{W^{k,p}} := \left(\int_\Omega \sum_{|\alpha| \leqslant k} |D^\alpha u|^p dA \right)^{1/p},$$

where $\alpha \in \mathbf{N}_0^2$. If $k = 0$, then $W^{k,p} = L^p$ and instead of $\|u\|_{L^p}$ we sometimes write $\|u\|_p$. Another Banach space $W_0^{k,p}(\Omega)$ arises by taking the closure of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$ (here $C_0^{k,p}(\Omega)$ is the space of k times continuously differentiable functions with compact support in Ω, [11, pp. 153–154]).

The main subject of this paper is a weak solution of Dirichlet's problem

$$\begin{cases} u_{z\bar{z}} = g(z), & z \in \Omega \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{1.1}$$

where $4u_{z\bar{z}} = \Delta u$ is the Laplacian of u. This is the Poisson's equation. A weak differentiable function u defined in a domain Ω with $u \in W_0^{1,p}(\Omega)$ is a weak solution of Poisson's equation if $D_1 u$ and $D_2 u$ are locally integrable in Ω, and

$$\int_\Omega (D_1 u D_1 v + D_2 u D_2 v + 4gv) dA = 0,$$

for all test functions $v \in C_0^1(\Omega)$.

It is well known that for $g \in L^p(\Omega)$, $p \geqslant 1$, the weak solution u of Poisson's equation is given explicitly as the sum of the Newtonian potential

$$N(g) = \frac{2}{\pi} \int_\Omega \log|z - w| g(w) dA(w),$$

and a harmonic function h such that $h|_{\partial\Omega} + N(g)|_{\partial\Omega} \equiv u|_{\partial\Omega}$. In particular, if $\Omega = \mathbf{U}$, then the function