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Bifurcating extremal domains for the first eigenvalue of the Laplacian

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Abstract

We prove the existence of a smooth family of non-compact domains $\Omega_s \subset \mathbb{R}^{n+1}$, $n \geqslant 1$, bifurcating from the straight cylinder $B^n \times \mathbb{R}$ for which the first eigenfunction of the Laplacian with 0 Dirichlet boundary condition also has constant Neumann data at the boundary: For each $s \in (-\varepsilon, \varepsilon)$, the overdetermined system

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega_s, \\ u = 0 & \text{on } \partial\Omega_s, \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \partial\Omega_s \end{cases}$$

has a bounded positive solution. The domains Ω_s are rotationally symmetric and periodic with respect to the $\mathbb{R}$-axis of the cylinder; they are of the form

$$\Omega_s = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \|x\| < 1 + s\cos\left(\frac{2\pi}{T_s}t\right) + O\left(s^2\right) \right\}$$

where $T_s = T_0 + O(s)$ and T_0 is a positive real number depending on n. For $n \geqslant 2$ these domains provide a smooth family of counter-examples to a conjecture of Berestycki, Caffarelli and Nirenberg. We also give rather precise upper and lower bounds for the bifurcation period T_0. This work improves a recent result of the second author.

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1. Introduction and main results

1.1. The problem

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and consider the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Denote by $\lambda_1(\Omega)$ the smallest positive constant λ for which this system has a solution (i.e. $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian on Ω with 0 Dirichlet boundary condition). By the Krein–Rutman theorem, the eigenvalue $\lambda_1(\Omega)$ is simple, and the corresponding eigenfunction (that is unique up to a multiplicative constant) has constant sign on Ω, see [14, Theorem 1.2.5]. One usually takes the eigenfunction u with $u > 0$ on Ω and $\int_\Omega u^2 = 1$. The eigenfunctions of higher eigenvalues must change sign on Ω, since they are orthogonal to the first eigenfunction. By the Faber–Krahn inequality,

$$\lambda_1(\Omega) \geqslant \lambda_1\big(B^n(\Omega)\big) \tag{2}$$

where $B^n(\Omega)$ is the round ball in $\mathbb{R}^n$ with the same volume as Ω. Moreover, equality holds in (2) if and only if $\Omega = B^n(\Omega)$, see [9] and [17]. In other words, round balls are minimizers for λ_1 among domains of the same volume. This result can also be obtained by reasoning as follows. Consider the functional $\Omega \to \lambda_1(\Omega)$ for all smooth bounded domains Ω in $\mathbb{R}^n$ of the same volume, say $\mathrm{Vol}(\Omega) = \alpha$. A classical result due to Garabedian and Schiffer asserts that Ω is a critical point for λ_1 (among domains of volume α) if and only if the first eigenfunction of the Laplacian in Ω with 0 Dirichlet boundary condition has also constant Neumann data at the boundary, see [11]. In this case, we say that Ω is an extremal domain for the first eigenvalue of the Laplacian, or simply an *extremal domain*. Extremal domains are then characterized as the domains for which the *overdetermined* system

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a positive solution (here ν is the outward unit normal vector field along $\partial\Omega$). By a classical result due to J. Serrin the only domains for which the system (3) has a positive solution are round balls, see [23]. One then checks that round balls are minimizers.

For domains with infinite volume, at first sight one cannot ask for "a domain that minimizes λ_1". Indeed, with $c\Omega = \{cz \mid z \in \Omega\}$ we have

$$\lambda_1(c\Omega) = c^{-2}\lambda_1(\Omega), \quad c > 0.$$